

Homological mirror symmetry is Fourier-Mukai transform

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Abstract

We interpret symplectic geometry as certain sheaf theory by constructing a sheaf of curved A_∞ algebras which in some sense plays the role of a “structure sheaf” for symplectic manifolds. An interesting feature of this “structure sheaf” is that the symplectic form itself is part of its curvature term. Using this interpretation homological mirror symmetry can be understood by well-known duality theories in mathematics: Koszul duality or Fourier-Mukai transform. In this paper we perform the above constructions over a small open subset inside the smooth locus of a Lagrangian torus fibration. In a subsequent work we shall use the language of derived geometry to obtain a global theory over the whole smooth locus. However we do not know how to extend this construction to the singular locus. As an application of the local theory we prove a version of homological mirror symmetry between a toric symplectic manifold and its Landau-Ginzburg mirror.

1. Introduction

1.1. Backgrounds and histories. Homological mirror symmetry conjecture was proposed by M. Kontsevich [18] in an address to the 1994 International Congress of Mathematicians, aiming to give a mathematical framework to understand the mirror phenomenon originated from physics. Roughly speaking this conjecture predicts a quasi-equivalence between two A_∞ triangulated categories $\mathrm{Fuk}(M)$ and $\mathrm{D}_{\mathrm{coh}}^b(M^\vee)$ naturally associated to a symplectic manifold M and a complex manifold M^\vee . Note that despite of the notation, the mirror manifold M^\vee is not uniquely determined by M .

After nearly two decades since Kontsevich’s proposal, his conjecture has been generalized and has been proven for lots of cases. We only mention a few of them here ¹. For the case of Elliptic curves this is due to A. Polishchuk and E. Zaslow [22]; for higher dimensional torus due to M. Kontsevich himself and Y. Soibelman [19]; for quartic hypersurfaces due to P. Seidel [23]; for general Calabi-Yau hypersurfaces due to the recent work of N. Sheridan [26]. We should also mention the work of M. Abouzaid [1], P. Seidel [24], D. Auroux, L. Katzarkov, D. Orlov [2], and B. Fang, C.-C. M. Liu, D. Treumann, E. Zaslow [14] for homological mirror symmetry on non-Calabi-Yau manifolds.

In spite of our increasing knowledge of this conjecture, less is understood about the mathematical reason behind it. Indeed it is not clear from a purely mathematical point of view why computations in symplectic geometry should match with completely different computations in algebraic/complex geometry. A more fundamental question is why should mathematicians be convinced that such a surprising duality between symplectic geometry and complex geometry should exist at all.

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¹Apology for not making a more complete list.

The first attempt towards a general mathematical theory to understand the mirror phenomenon is given by A. Strominger, S-T. Yau and E. Zaslow. In [27] they proposed a geometric picture to produce mirror pairs: mirror duality should arise between (special) Lagrangian torus fibrations and the associated dual fibrations. We note that for purposes in this paper we do not need to have the specialness condition. To get interesting symplectic manifolds and complex manifolds (other than torus) one has to allow these fibrations to have singular fibers for which we refer to M. Gross and B. Siebert's theory on toric degenerations [15], [16].

While this gives a nice theory to produce mirror pairs, it gives less hint on why mirror pairs produced in this way should interchange symplectic geometry with complex geometry². Moreover it is not clear how homological mirror symmetry should arise between dual torus fibrations. The main purpose of the current paper is to answer these questions. Namely we show the mirror duality between symplectic geometry of a Lagrangian torus fibration and complex geometry of its dual fibration is in fact a well-known duality in mathematics: Koszul duality or its global version Fourier-Mukai transform.

The paper grew out of understanding an algebraic framework for K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono's series of papers [11],[12],[13]. There are also inspiring works of Fukaya [8] and Seidel [25].

1.2. An example. We begin with a simple example illustrating the main ideas. Let \mathbb{R}^\vee be endowed with a linear coordinate x^\vee , and let \mathbb{R} be endowed with the dual coordinate y (this choice of notation will be clear later). The cotangent bundle $T^*(\mathbb{R}^\vee) = \mathbb{R}^\vee \times \mathbb{R}$ has a canonical symplectic form $\omega := dx^\vee \wedge dy$. To have a Lagrangian torus fibration we consider the quotient space $M := \mathbb{R}^\vee \times (\mathbb{R}/\mathbb{Z})$. Since ω is translation invariant, it descends to a symplectic form on M . The projection map $\pi : M \rightarrow \mathbb{R}^\vee$ defines a Lagrangian torus fibration.

Consider a complex vector bundle over \mathbb{R}^\vee whose fiber over a point $u \in \mathbb{R}^\vee$ is the cohomology group $H^*(\pi^{-1}(u), \mathbb{C})$. Denote by \mathcal{H} the corresponding sheaf of C^∞ -sections. It is well-known that the sheaf \mathcal{H} is in fact a D-module endowed with the Gauss-Manin connection ∇ . Moreover the cup product on cohomology defines an algebra structure on \mathcal{H} which is ∇ -flat. Thus the de Rham complex $\Omega^*(\mathcal{H})$ with coefficients in the D-module \mathcal{H} has a differential graded (commutative) algebra structure. We remark that the construction of this differential graded algebra does not involve the symplectic structure on M .

To encode the symplectic structure into the algebra $\Omega^*(\mathcal{H})$, we can add a curvature term to it which is just the symplectic form ω itself (up to sign)! More precisely if we denote by $e := dy$ a ∇ -flat integral generator for \mathcal{H}^1 (degree one part). Then $-\omega$ can be viewed as an element of $\Omega^*(\mathcal{H})$ by writing it as $e \otimes dx^\vee$. Note that this element $-\omega$ is of even degree, and it is closed in $\Omega^*(\mathcal{H})$. Moreover since the differential graded algebra $\Omega^*(\mathcal{H})$ is supercommutative any such element can be viewed as a curvature term. Thus we have obtained a curved differential graded algebra structure on $\Omega^*(\mathcal{H})$ which will be denoted by \mathcal{O}_M^ω in the following. We think of it as the structure sheaf of "symplectic functions" on $M = \mathbb{R}^\vee \times (\mathbb{R}/\mathbb{Z})$.

1.3. From symplectic functions to holomorphic functions: Koszul duality. To understand the mirror phenomenon which interchanges symplectic geometry with complex geometry, let us compute the Koszul dual algebra of for \mathcal{O}_M^ω . We shall work over the base ring $\Omega_{\mathbb{R}^\vee}^*$, the complex valued de Rham complex of the base manifold \mathbb{R}^\vee . If we ignore the curvature term $-\omega = e \otimes dx^\vee$, the underlying differential graded algebra structure of \mathcal{O}_M^ω is simply the

²The original SYZ explanation for this was from the so-called T-duality in physics.

exterior algebra generated by e over $\Omega_{\mathbb{R}^\vee}^*$. Hence its Koszul dual algebra is the symmetric algebra $\text{sym}_{\Omega_{\mathbb{R}^\vee}^*}(\mathbf{y}^\vee)$ generated by an even variable $\mathbf{y}^\vee := e^\vee[-1]$.

For the curvature term observe that $-\omega = e \otimes d\mathbf{x}^\vee$ is a linear curvature since we are working over $\Omega_{\mathbb{R}^\vee}^*$. It is well-known in Koszul dual theory how to deal with linear curvatures: they introduce additional differential in Koszul dual algebras. The resulting Koszul dual algebra of \mathcal{O}_M^ω is thus the symmetric algebra $\text{sym}_{\Omega_{\mathbb{R}^\vee}^*}(\mathbf{y}^\vee) \cong \text{sym}(\mathbf{y}^\vee) \otimes \Omega_{\mathbb{R}^\vee}^*$ endowed with a differential acting on it as the operator $(\partial_{\mathbf{x}^\vee} + \partial_{\mathbf{y}^\vee})d\mathbf{x}^\vee$. If we perform a change of variable by

$$\mathbf{x}^\vee \mapsto \mathbf{x}^\vee, \quad \mathbf{y}^\vee \mapsto -\sqrt{-1}\mathbf{y}^\vee, \quad d\mathbf{x}^\vee \mapsto d\bar{\mathbf{z}};$$

this Koszul dual algebra is identified with the Dolbeault algebra resolving the structure sheaf of the complex manifold $\mathbb{T}\mathbb{R}^\vee = \mathbb{R}^\vee \times \mathbb{R}^\vee$ with its canonical complex structure. From this example we see that the algebraic reason that Koszul duality interchanges a linear curvature term with an additional differential in its Koszul dual, when applied to this situation, gives a direct link between the symplectic structure and its mirror complex structure.

1.4. Approaching HMS via duality of algebras. From the previous example we see that the sheaf of symplectic functions is “Koszul dual” to the sheaf of holomorphic functions. In general we propose to understand the homological mirror symmetry conjecture in the following steps:

- I. Associated to any Lagrangian torus fibration $\pi : M \rightarrow B$ construct a sheaf of A_∞ algebras \mathcal{O}_M^ω which plays the role of a structure sheaf in symplectic geometry;
- II. Let $\pi^\vee : M^\vee \rightarrow B$ be the dual torus fibration ³, construct another sheaf $\mathcal{O}_{M^\vee}^{\text{hol}}$ over B , which in some sense is “Koszul dual” to \mathcal{O}_M^ω ;
- III. Associated to any Lagrangians (with unitary local systems) in M construct a module over \mathcal{O}_M^ω , and show that the Fukaya category $\text{Fuk}(M)$ fully faithfully embeds into the category of modules over \mathcal{O}_M^ω ;
- IV. Understand the module correspondence between the “Koszul dual” algebras \mathcal{O}_M^ω and $\mathcal{O}_{M^\vee}^{\text{hol}}$.

The current paper contains partial results in all four steps mentioned above. Here “partial” mainly means that we work out the constructions over a small local open subset inside the smooth locus B^{int} of a Lagrangian torus fibration. The notion of a sheaf of A_∞ algebras needs more explanation: locally over small open subsets in B^{int} we can construct honest sheaves of A_∞ algebras while globally we expect these local data can be glued up to homotopy, producing a homotopy sheaf of A_∞ algebras. In this paper we will be mainly concerned with local constructions except in Section 7. Global constructions will appear in a forthcoming work [28]. We should also confess that it is not clear at present how the singular locus B^{sing} should enter into this study.

There are lots of advantages in this new approach to homological mirror symmetry conjecture. The first advantage is a natural construction of (local) mirror functors (see Sections 4 and 5) using classical Koszul duality theory of modules. Note that functors constructed this way are A_∞ functors with explicit formulas. It is also conceptually clearer in this approach how

³For this we should refer to M. Gross and B. Siebert’s constructions [15], [16].

lots of ad-hoc constructions in mirror symmetry should enter into the theory. For instance the inclusion of a B-field to complexify symplectic moduli, or the appearance of quantum corrections in mirror constructions. Finally we believe this approach offers a way to prove homological mirror symmetry conjecture in an abstract form (i.e. without computing both sides explicitly).

In the remaining part of the introduction we give an overview of materials contained in each section.

1.5. Section 2. In this section we deal with step I over a small open subset $\mathcal{U} \subset B^{\text{int}}$. Denote by $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ the projection map.

Consider the sheaf \mathcal{H} of C^∞ sections of $R\pi_* \Lambda^\pi \otimes C_{\mathcal{U}}^\infty$ where Λ^π is certain relative Novikov ring (see Section 2 for its precise definition). The Bott-Morse Lagrangian Floer theory developed in [9] and [7] endows for each Lagrangian fiber L_u an A_∞ algebra structure on $H^*(L_u, \Lambda^\pi)$. For our purpose we need to consider this A_∞ structure as a family over \mathcal{U} . This construction has been taken care of by K. Fukaya in [7] which we follow in this paper.

Just as in the one dimensional example, the sheaf \mathcal{H} is a D-module with the Gauss-Manin connection. But this time the structure maps m_k are not ∇ -flat in general. Our main observation is the following compatibility between the D-module structure on \mathcal{H} and its A_∞ structure.

$$[\nabla, m_k] = \sum_{i=1}^{k+1} m_{k+1}(\cdots, \omega, \cdots)$$

This equation is a simple consequence of cyclic symmetry proved in [7]. An immediate corollary of these compatibility equations is the following theorem.

1.6. Theorem. *There is a curved A_∞ algebra structure on the de Rham complex $\Omega_{\mathcal{U}}^*(\mathcal{H})$ whose curvature is given by $m_0 - \omega$.*

We denote this sheaf of curved A_∞ algebras over \mathcal{U} by $\mathcal{O}_{M(\mathcal{U})}^{\omega, \text{can}}$ (or simply $\mathcal{O}^{\omega, \text{can}}$). The notation is due to the usage of the A_∞ algebras $H^*(L_u, \Lambda^\pi)$ which were called canonical models in [10].

One can replace the canonical model $H^*(L_u, \Lambda^\pi)$ by the full de Rham complex $\Omega^*(L_u, \Lambda^\pi)$ on each fiber L_u . The same construction also works in this case (in fact in Section 2 we mainly work with this version) if we replace the sheaf \mathcal{H} by the relative de Rham complex of π . We denote the resulting sheaf of A_∞ algebras by $\mathcal{O}_{M(\mathcal{U})}^\omega$ (or simply \mathcal{O}^ω). There is a subtle difference between the two sheaves of A_∞ algebras \mathcal{O}^ω and $\mathcal{O}^{\omega, \text{can}}$ in view of mirror symmetry which is explained in Sections 4 and 5. Roughly the “mirror” of the $\mathcal{O}^{\omega, \text{can}}$ is the tangent bundle $T\mathcal{U}$ while that of \mathcal{O}^ω is the dual torus bundle over \mathcal{U} .

1.7. Section 3. We show how to obtain modules over the sheaf of symplectic functions (\mathcal{O}^ω or $\mathcal{O}^{\omega, \text{can}}$) from Lagrangian branes. In this paper we only deal with Lagrangian branes of the form (L_u, α) where L_u is a Lagrangian torus fiber over a point $u \in \mathcal{U}$ and α is a purely imaginary one form in $H^1(L_u, \mathbb{C})$. Denote by $\text{Fuk}^\pi(M)$ the full subcategory of $\text{Fuk}(M)$ consisting of such Lagrangian branes.

1.8. Theorem. *There exists a linear A_∞ functor $P : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}^\omega)$ which is a quasi-equivalence onto its image. The same statement also holds for $\mathcal{O}^{\omega, \text{can}}$.*

We should mention that in this section we work with the following assumption.

Weak unobstructedness assumption: *For any $u \in \mathcal{U}$, the m_0 term of the curved A_∞ algebra on $H^*(L_u, \Lambda^\pi)$ is a scalar multiple of $\mathbf{1}$, the strict unit of $H^*(L_u, \Lambda^\pi)$.*

1.9. Section 4. In this section we study mirror symmetry by Koszul duality theory. First we define the Koszul dual algebra of the sheaf $\mathcal{O}^{\omega, \text{can}}$. Let us first describe this Koszul dual algebra. We continue to work with the above assumption. This assumption implies that for any $\mathbf{b} \in H^1(L_u, \Lambda^\pi)$ we have

$$m_0 + m_1(\mathbf{b}) + m_2(\mathbf{b}, \mathbf{b}) + m_3(\mathbf{b}, \mathbf{b}, \mathbf{b}) + \cdots + m_k(\mathbf{b}, \dots, \mathbf{b}) + \cdots \equiv 0 \pmod{1}$$

where 1 is the strict unit for the A_∞ algebra $H^*(L_u, \Lambda^\pi)$. Then as in [11] we can define a potential function W on the tangent bundle $\mathbb{T}\mathbb{U}$ by

$$W(\mathbf{u}, \mathbf{b}) := \text{the coefficient of 1 of the sum } \sum_{k=0}^{\infty} m_k(\mathbf{b}^{\otimes k}).$$

We then show that that W is in fact a “holomorphic” function on $\mathbb{T}\mathbb{U}$ with its natural complex structure. Define $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ to be the “Dolbeault complex” of the structure sheaf of $\mathbb{T}\mathbb{U}$ endowed with a curvature term given by W ⁴. Then $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ is Koszul dual to $\mathcal{O}_{M(\mathbb{U})}^{\omega, \text{can}}$ in the following sense.

1.10. Lemma. *There is a universal Maurer-Cartan element τ in the tensor product curved A_∞ algebra $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes_{\Omega_u^*} \mathcal{O}_{M(\mathbb{U})}^{\omega, \text{can}}$.*

This lemma can be used to construct an A_∞ functor $\Phi^\tau : \text{tw}(\mathcal{O}^{\omega, \text{can}}) \rightarrow \text{tw}(\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}})$ which is also a quasi-equivalence onto its image. Pre-composing with the functor P in Theorem 1.8 yields the following.

1.11. Theorem. *The composition $\text{Fuk}^\pi(M) \xrightarrow{P} \text{tw}(\mathcal{O}^{\omega, \text{can}}) \xrightarrow{\Phi^\tau} \text{tw}(\mathcal{O}^{\text{hol}})$ is a quasi-equivalence onto its image.*

We can also identify the image of this composition functor under certain assumptions. These assumptions are automatic for a Calabi-Yau manifold and a special Lagrangian submanifold.

1.12. Theorem. *Assume that strictly negative Maslov index does not contribute to structure maps m_k , and assume further that the potential function $W = 0$. Then the object $\Phi^\tau(\mathcal{L}_u(\alpha))$ is quasi-isomorphic to a skyscraper sheaf $\Lambda^\pi(u, \alpha)$ supported at $u \in \mathbb{U}$.*

1.13. Section 5. This section is parallel to the previous section, but replacing Koszul duality with Fourier-Mukai transform. Similar results as Theorem 1.11 and Theorem 1.12 are obtained in the case of \mathcal{O}^ω . The universal Maurer-Cartan element τ in Theorem 1.10 is replaced by a quantum version of the Poincaré bundle. Again this construction yields (local) A_∞ mirror functors. We refer to Section 5 for more details.

1.14. Section 6. As an application of the general theory, we prove a version of homological mirror symmetry between a compact toric symplectic manifold and its Landau-Ginzburg mirror. We summarize the main results in the following theorem.

1.15. Theorem. *Let M be a compact smooth toric symplectic manifold, and denote by $\pi : M(\Delta^{\text{int}}) \rightarrow \Delta^{\text{int}}$ the Lagrangian torus fibration over the interior of the polytope of M . Then there exists an A_∞ functor $\Psi : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}_{T(\Delta^{\text{int}})}^{\text{hol}})$ which is a quasi-equivalence onto its image. If furthermore M is Fano of complex dimension less or equal to two, then we can*

⁴Here several words are in quote since we need to work with Novikov ring, and hence these notions need to be interpreted correctly.

work over \mathbb{C} instead of over Λ^π . Moreover in this case the reduction of Ψ over \mathbb{C} is a quasi-equivalence.

Remark: The Fano assumption is to ensure convergence over \mathbb{C} while the dimension assumption is more of technical nature. We expect the functor Ψ to be always essentially surjective over Novikov ring as long as W has isolated singularities.

1.16. Section 7. Materials in this section are mainly speculative. We propose a conjectural framework to study global aspects of mirror symmetry. The theory of derived geometry naturally enter into this framework. We conjecture that local symplectic functions constructed in this paper satisfy homotopy descend condition to obtain a homotopy sheaf of A_∞ algebras. This property is essential to understand instanton corrections in the mirror manifold construction. Details of these global constructions will appear in [28].

1.17. Appendices A and B. We include materials on homological algebras of A_∞ modules over curved A_∞ algebras which might have “internal curvatures”. We also interpret Koszul duality functors as an affine version of Fourier-Mukai transform. Materials in these appendices are well-known to experts, but are not easy to find in literature. We include them here for completeness. A detailed explanation of the sign conventions used in this paper is also included in the appendix B.

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1.19. Notations and Conventions.

- Λ^π : relative Novikov ring, see Definition 2.7;
- $\text{Fuk}^\pi(M)$: the full A_∞ subcategory of $\text{Fuk}(M)$ consisting of Lagrangian torus fibers endowed with purely imaginary invariant one form;
- e_1, \dots, e_d : a ∇ -flat basis for the integral lattice $R^1\pi_*\mathbb{Z}$ over \mathcal{U} ;
- $x_1^\vee, \dots, x_d^\vee$: the corresponding of (action) affine coordinates on \mathcal{U} ;
- y_1, \dots, y_d : angle coordinates on Lagrangian torus fibers;
- $y_1^\vee, \dots, x_d^\vee$: dual coordinates on dual torus fibers.
- We use the following two signs frequently in this paper.

$$(1) \ \epsilon_k := \sum_{i=1}^{k-1} (k-i)|a_i| \text{ associated to homogeneous tensor products } a_1 \otimes \dots \otimes a_k;$$

$$(2) \ \eta_k = \sum_{i=1}^{k-1} |a_i|(|b_{i+1}| + \dots + |b_k|) \text{ resulting from the permutation}$$

$$(a_1 \otimes b_1) \otimes \dots \otimes (a_k \otimes b_k) \mapsto (a_1 \otimes \dots \otimes a_k) \otimes (b_1 \otimes \dots \otimes b_k).$$

We will use two different sign conventions: one used by the authors of [9] in Lagrangian Floer theory, and the other one as in [17]. Structure maps of the former will be denoted by m_k , and the latter by m_k^ϵ . They are related by

$$m_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{\epsilon_k} m_k^\epsilon(a_1 \otimes \cdots \otimes a_k).$$

2. Symplectic functions

Let $\pi : M(U) \rightarrow U$ be a small (to be made precise below) and smooth local piece of a Lagrangian torus fibration $M \rightarrow B$. In this section we present a construction of a sheaf of curved A_∞ algebras over U encoding symplectic geometry of M . An interesting feature of our construction is that the symplectic form ω itself enters as part of the curvature term.

2.1. A_∞ algebras associated to Lagrangian submanifolds. Let L be a relatively spin compact Lagrangian submanifold in a symplectic manifold (M, ω) . In [9] and [7] a curved A_∞ algebra structure was constructed on $\Omega^*(L, \Lambda_0)$, the de Rham complex of L with coefficients in certain Novikov ring Λ_0 over \mathbb{C} ⁵. Here coefficient ring Λ_0 is defined by

$$\Lambda_0 := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}^{\geq 0}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\}$$

where T is a formal parameter of degree zero. Note that the map $\text{val} : \Lambda_0 \rightarrow \mathbb{R}$ defined by $\text{val}(\sum_{i=1}^{\infty} a_i T^{\lambda_i}) := \inf_{a_i \neq 0} \lambda_i$ endows Λ_0 with a valuation ring structure. Denote by Λ_0^+ the subset of Λ_0 consisting of elements with strictly positive valuation. This is the unique maximal ideal of Λ_0 . In [9] there was an additional parameter e to encode Maslov index to have a \mathbb{Z} -graded A_∞ structure. If we do not use this parameter, we need to work with $\mathbb{Z}/2\mathbb{Z}$ -graded A_∞ algebras.

We briefly recall the construction of this A_∞ structure on $\Omega^*(L, \Lambda_0)$. Let $\beta \in \pi_2(M, L)$ be a class in the relative homotopy group, and choose an almost complex structure J compatible with (or simply tamed) ω . Form $\mathcal{M}_{k+1, \beta}(M, L; J)$, the moduli space of stable $(k+1)$ -marked J -holomorphic disks in M with boundary lying in L of homotopy class β with suitable regularity condition in interior and on the boundary. The moduli space $\mathcal{M}_{k+1, \beta}(M, L; J)$ is of virtual dimension $d + k + \mu(\beta) - 2$ (here $\mu(\beta)$ is the Maslov index of β).

There are $(k+1)$ evaluation maps $ev_i : \mathcal{M}_{k+1, \beta}(M, L; J) \rightarrow L$ for $i = 0, \dots, k$ which can be used to define a map $m_{k, \beta} : (\Omega^*(L, \mathbb{C})^{\otimes k}) \rightarrow \Omega^*(L, \mathbb{C})$ of form degree $2 - \mu(\beta) - k$ by formula

$$m_{k, \beta}(\alpha_1, \dots, \alpha_k) := (ev_0)_!(ev_1^* \alpha_1 \wedge \cdots \wedge ev_k^* \alpha_k).$$

To get an A_∞ algebra structure we need to combine $m_{k, \beta}$ for different β 's. For this purpose we define a submonoid $G(L)$ of $\mathbb{R}^{\geq 0} \times 2\mathbb{Z}$ as the minimal one generated by the set

$$\left\{ \left(\int_{\beta} \omega, \mu(\beta) \right) \in \mathbb{R}^{\geq 0} \times 2\mathbb{Z} \mid \beta \in \pi_2(M, L), \mathcal{M}_{0, \beta}(M, L; J) \neq \emptyset \right\}.$$

⁵Strictly speaking this version of Floer theory was developed by K. Fukaya in [7] heavily based on the work of himself, Y.-G. Oh, H. Ohta and K. Ono [9].

Then we can define the structure maps $m_k : (\Omega(L, \Lambda_0))^{\otimes k} \rightarrow \Omega(L, \Lambda_0)$ by

$$m_k(\alpha_1, \dots, \alpha_k) := \sum_{\beta \in G(L)} m_{k,\beta}(\alpha_1, \dots, \alpha_k) T^{\int_\beta \omega}.$$

Note that we need to use the Novikov coefficients here since the above sum might not converge for a fixed value of T . The boundary stratas of $\mathcal{M}_{k+1,\beta}(M, L; J)$ are certain fiber products of the diagram

$$\begin{array}{ccc} \mathcal{M}_{i+1,\beta_1}(M, L; J) \times_L \mathcal{M}_{j+1,\beta_2}(M, L; J) & \longrightarrow & \mathcal{M}_{j+1,\beta_2}(M, L; J) \\ \downarrow & & \downarrow \text{ev}_l \\ \mathcal{M}_{i+1,\beta_1}(M, L; J) & \xrightarrow{\text{ev}_0} & L. \end{array}$$

Here $1 \leq l \leq j$, $i + j = k + 1$, and $\beta_1 + \beta_2 = \beta$. Indeed using this description of the boundary stratas the A_∞ axiom for structure maps m_k is an immediate consequence of Stokes formula.

We should emphasize that a mathematically rigorous realization of the above ideas involves lots of delicate constructions. Indeed the moduli spaces $\mathcal{M}_{k+1,\beta}(M, L; J)$ are not smooth manifolds, but Kuranishi orbifolds with corners, which causes trouble to define an integration theory. Even if this regularity problem is taken of there are still transversality issues to define maps $m_{k,\beta}$ to have the expected dimension. Moreover it is not enough to take care of each individual moduli space since the A_∞ relations for m_k follows from analyzing the boundary stratas in $\mathcal{M}_{k+1,\beta}(M, L; J)$. Thus one needs to prove transversality of evaluation maps that are compatible for all k and β . Furthermore one also need to deal with not only disk bubbles, but also sphere bubbles and regularity and transversality issues therein. We refer to the original constructions of [9] and [7] for solutions of these problems.

2.2. Main properties of the A_∞ algebra $\Omega^*(L, \Lambda_0)$. Let us summarize some of the main properties of $\Omega^*(L, \Lambda_0)$ proved in [9] and [7].

- (Invariants of symplectic geometry) The homotopy type of this A_∞ structure on $\Omega(L, \Lambda_0)$ is independent of J , moreover it is invariant under symplectomorphism;
- (Deformation property) This A_∞ structure is a deformation of the classical differential graded algebra structure on the de Rham complex with coefficients in Λ_0 ;
- (Algebraic property) The A_∞ structure can be constructed to be strict unital and cyclic so that constant function $\mathbf{1}$ is the strict unit and structure maps m_k are cyclic with respect to the Poincaré pairing $\langle \alpha, \beta \rangle = \int_L \alpha \wedge \beta$.

We shall describe one more important property of the A_∞ algebra associated to L . This is analogous to the divisor equation in Gromov-Witten theory of closed Riemann surfaces. Such a generalization was first observed by C.-H. Cho [4] in the case of Fano toric manifolds. Later in [7] Cho's result was generalized by K. Fukaya to general symplectic manifolds.

2.3. Lemma. [K. Fukaya [7] Lemma 13.1 and 13.2] Let $b \in H^1(L, \Lambda_0)$ and consider any lift of it to an element of $\Omega^1(L, \Lambda_0)$ which we still denote by b . Then for any $k \geq 0$ and $l \geq 0$ we have

$$\sum_{l_0 + \dots + l_k = l} m_{k+l,\beta}(b^{\otimes l_0}, \alpha_1, \dots, b^{\otimes l_{k-1}}, \alpha_k, b^{\otimes l_k}) = \frac{1}{l!} \langle b, \partial \beta \rangle^l m_{k,\beta}(\alpha_1, \dots, \alpha_k).$$

Here $\partial\beta \in H^1(L, \mathbb{Z})$ is the boundary of β .

The main construction of [7] was devoted to constructing compatible Kuranishi structure and continuous multi-section perturbations that are compatible with forgetful maps between Kuranishi spaces $\mathcal{M}_{k+l+1, \beta}(M, L; J) \rightarrow \mathcal{M}_{k+1, \beta}(M, L; J)$ which forget the last l marked points for various k and l . Indeed with this structure being taken care of the above lemma is a simple exercise on iterated integrals.

2.4. Local family of A_∞ algebras. If $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ is a local smooth family of Lagrangian torus, we get an A_∞ algebra for each point $u \in \mathcal{U}$. We would like to consider this as giving us a family of A_∞ algebra over \mathcal{U} . However there is a delicate point involved here: in general this fiber-wise construction does not produce an A_∞ algebra over \mathcal{U} on the relative de Rham complex of π even for a generic almost complex structure J due to wall-crossing discontinuity. In [7] Section 13 K. Fukaya constructed such a family by allowing almost complex structures to depend on the Lagrangians. Let us briefly recall Fukaya's construction here.

Fix a point $0 \in \mathcal{U}$ and let $u \in \mathcal{U}$ be a point in a small neighborhood \mathcal{U} of 0 , the corresponding near-by Lagrangian torus fiber L_u can be viewed as the graph of a real one form $\sum_{i=1}^n c_i e_i \in H^1(L_0, \mathbb{R})$ in the cotangent bundle of L_0 . Then one chooses a diffeomorphism $F_u : M \rightarrow M$ supported in $M(\mathcal{U})$ such that $F_u(L_0) = L_u$ and the almost complex structure $J_u := (F_u)_* J_0$ is tamed by the symplectic form ω . The main advantage of this choice of almost complex structures depending on Lagrangians is that it induces identification of various moduli spaces:

$$(F_u)_* : \mathcal{M}_{k, \beta}(M, L_0; J_0) \cong \mathcal{M}_{k, (F_u)_* \beta}(M, L_u; J_u).$$

Thus maps $m_{k, \beta}$ involved in the definition of the A_∞ algebra associated a Lagrangian submanifold L_u ($u \in \mathcal{U}$) does not depend on the base parameter u . Hence the structure maps $m_k := \sum_{\beta} m_{k, \beta} T^{\int_{\beta} \omega}$ depend on the u -parameter only via symplectic area $\int_{\beta} \omega$. The follow lemma makes this dependence explicit.

2.5. Lemma. *Let L_u be a near-by fiber of L_0 such that L_u is defined as the graph of the one form $\alpha_u := \sum_{i=1}^n c_i e_i \in H^1(L_0, \mathbb{R})$. Then for each $\beta \in \pi_2(M, L_0)$ we have*

$$\int_{(F_u)_* \beta} \omega - \int_{\beta} \omega = \langle \alpha_u, \partial\beta \rangle$$

where $\partial\beta \in \pi_1(L_0)$ is the boundary of β .

Proof. This is an exercise in Stokes' formula. □

2.6. Relative Novikov ring. To have a sheaf of A_∞ algebras over \mathcal{U} we need to introduce another Novikov type coefficient ring. Let $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ be a small smooth local piece of Lagrangian torus fibration as above, and let $0 \in \mathcal{U}$ be the base point in \mathcal{U} . The family of monoids $G(L_u)$ defines a bundle of monoids over \mathcal{U} . We denote this bundle by G , and write $\beta \in G$ to mean a continuous section of F over \mathcal{U} . For instance given an element $\beta \in G(L_0)$ we get a continuous section of G by assigning $u \mapsto (F_u)_* \beta$.

2.7. Definition. *The relative Novikov ring Λ^π associated to the family of Lagrangians $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ is defined by*

$$\Lambda^\pi := \left\{ \sum_{i=1}^{\infty} a_i T^{\beta_i} \mid a_i \in \mathbb{C}, \beta_i \in G; \# \left\{ a_i \mid a_i \neq 0, \int_{\beta_i} \omega|_{u=0} \leq E \right\} < \infty, \forall E \in \mathbb{R} \right\}.$$

The ring Λ^π is a \mathbb{Z} -graded ring with T^β of degree $\mu(\beta)$. Note that the Maslov index map μ is well-defined on G since it is preserved under diffeomorphisms F_u . This grading makes the operator $m_{k,\beta}T^\beta$ homogeneous of degree $2-k$.

The ring Λ^π can also be endowed with a valuation by evaluating symplectic area at the base point $0 \in \mathcal{U}$, i.e.

$$\text{val}\left(\sum_{i=1}^{\infty} \alpha_i T^{\beta_i}\right) := \inf_{\alpha_i \neq 0} \int_{\beta_i} \omega|_{u=0}.$$

We can define a decreasing filtration on Λ^π by setting $F^{\leq E} : \text{val}^{-1}([E, \infty))$. This filtration is called energy filtration. It is a useful tool since it induces a spectral sequence to compute Floer homology, see Chapter 6 of [9].

2.8. A sheaf of A_∞ algebras. Lagrangian Floer theory developed in [9] and [7] can be formulated over the ring Λ^π . Its relationship to the ring Λ_0 is there is a ring homomorphism $\Lambda^\pi \rightarrow \Lambda_0$ for each point $u \in \mathcal{U}$ defined by

$$\sum_{i=1}^{\infty} \alpha_i T^{\beta_i} \mapsto \sum_{i=1}^{\infty} \alpha_i T^{\int_{\beta_i} \omega}$$

where we consider β_i as an element of $G(L_u)$.

Using this Novikov ring we can define a sheaf of A_∞ algebra structure on $\Omega_\pi(\Lambda^\pi)$, the relative de Rham complex with coefficients in Λ^π . Explicitly an element of $\Omega_\pi(\Lambda^\pi)$ is of the form $\sum_{i=1}^{\infty} \alpha_i T^{\beta_i}$ satisfying the same finiteness condition as in the definition of Λ^π . Here $\alpha_i \in \Omega_\pi(\mathbb{C})$ are \mathbb{C} -valued relative differential forms.

The structure maps of this A_∞ algebra $\Omega_\pi(\Lambda^\pi)$ are defined by

$$m_k(\alpha_1, \dots, \alpha_k) := \sum_{\beta \in G} m_{k,\beta}(\alpha_1, \dots, \alpha_k) T^\beta,$$

and we extend these maps Λ^π -linearly to all elements of $\Omega_\pi(\Lambda^\pi)$.

2.9. D-module structure on $\Omega_\pi(\Lambda^\pi)$. Observe that the sheaf $\Omega_\pi(\mathbb{C})$, being the relative de Rham complex with complex coefficients, has a D-module over \mathcal{U} . We can extend this D-module structure to $\Omega_\pi(\Lambda^\pi)$ by

$$\nabla(T^\beta) := -\nabla\left(\int_{(F_u)_*\beta} \omega\right) T^\beta \quad (2.1)$$

and Leibniz rule⁶. We denote this derivation by ∇ , and call it the Gauss-Manin connection.

2.10. Variational structure on $\Omega_\pi(\Lambda^\pi)$. Our next goal is to study the variational structure of the A_∞ structure m_k using the Gauss-Manin connection ∇ . For this we consider the de Rham complex of the D-module $\Omega_\pi(\Lambda^\pi)$ over \mathcal{U} . As a sheaf over \mathcal{U} this is the same as $\Omega_\pi(\Lambda^\pi) \otimes_{C_\mathcal{U}^\infty} \Omega_\mathcal{U}^*$. We wish to extend m_k on $\Omega_\pi(\Lambda^\pi)$ to this tensor product. For this purpose it is more convenient to use a different sign convention which is better to form tensor products.

⁶This definition is due to the fact that in the convergent case we specialize T to be e^{-1} , as is done in [5] Section 13.

We refer to the new sign convention as the ϵ sign convention. Structure maps in this sign convention will be denoted by m_k^ϵ . It is related to the previous sign convention by formula

$$m_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{\epsilon_k} m_k^\epsilon(a_1 \otimes \cdots \otimes a_k)$$

where $\epsilon_k := \sum_{i=1}^{k-1} (k-i)|a_i|$. We extend the maps m_k^ϵ on $\Omega_\pi(\Lambda^\pi)$ to its de Rham complex to get maps $m_k^\epsilon : (\Omega_\pi(\Lambda^\pi) \otimes \Omega_U^*)^k \rightarrow \Omega_\pi(\Lambda^\pi) \otimes \Omega_U^*$ which are defined by

$$\begin{aligned} m_0^\epsilon &:= m_0 \otimes 1; \\ m_1^\epsilon(f \otimes \alpha) &:= m_1^\epsilon(f) \otimes \alpha + (-1)^{|f|} f \otimes d_{dR} \alpha; \\ m_k^\epsilon((f_1 \otimes \alpha_1) \otimes \cdots \otimes (f_k \otimes \alpha_k)) &:= (-1)^{\eta_k} m_k^\epsilon(f_1, \dots, f_k) \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k) \end{aligned}$$

where the sign is given by $\eta_k = \sum_{i=1}^{k-1} |\alpha_i|(|f_{i+1}| + \cdots + |f_k|)$. Here we have abused the notation m_k^ϵ , but no confusion should arise. It is straightforward to check that the maps m_k^ϵ defines an A_∞ algebra structure on the tensor product $\Omega_\pi(\Lambda^\pi) \otimes \Omega_U^*$.

2.11. Lemma. *For all $k \geq 0$ we have the following compatibility between the A_∞ structure and the D -module structure on $\Omega_\pi(\Lambda^\pi)$:*

$$[\nabla, m_k^\epsilon] = \sum_{i=1}^{k+1} (-1)^{i-1} m_{k+1}^\epsilon(\text{id}^{i-1} \otimes \omega \otimes \text{id}^{k-i+1}). \quad (2.2)$$

Here ω is the symplectic form of M restricted to $M(U)$, and it is viewed as an element of $\Omega_\pi(\Lambda^\pi) \otimes_{\mathbb{C}U} \Omega_U^*$. Locally in action-angle coordinates $\omega = \sum_{i=1}^n -dy_i \otimes dx_i^\vee = \sum_{i=1}^n -e_i \otimes dx_i^\vee$.

Proof. Up to signs this lemma is a direct consequence of Lemma 2.5 and Lemma 2.3. We include the proof here to illustrate our sign conventions. It is enough to prove the lemma for flat sections $f_1, \dots, f_k \in \Omega_\pi(\Lambda^\pi)$. The left hand side operator applied to $f_1 \otimes \cdots \otimes f_k$ gives

$$\begin{aligned} \nabla m_k^\epsilon(f_1 \otimes \cdots \otimes f_k) &= (-1)^{|f_1| + \cdots + |f_k| + 2 - k} \sum_{\beta} m_{k,\beta}^\epsilon(f_1 \otimes \cdots \otimes f_k) \nabla(T^\beta) \\ &= (-1)^{|f_1| + \cdots + |f_k| + 2 - k} (-1)^{\epsilon_k} \sum_{\beta} m_{k,\beta}(f_1 \otimes \cdots \otimes f_k) T^\beta \otimes \nabla\left(-\int_{\beta} \omega\right) \\ &= (-1)^{|f_1| + \cdots + |f_k| + 1 - k} (-1)^{\epsilon_k} \cdot \sum_{\beta} m_{k,\beta}(f_1 \otimes \cdots \otimes f_k) \langle \partial\beta, e_i \rangle T^\beta \otimes dx_i^\vee \quad (\text{by Lemma 2.5}) \\ &= \sum_{\beta, 1 \leq j \leq k+1} (-1)^{|f_1| + \cdots + |f_k| + 1 - k} (-1)^{\epsilon_k} (-1)^{|f_1|k + \cdots + |f_{j-1}|(k-j+2) + (k-j+1) + \cdots + |f_{k-1}|} \cdot m_{k+1,\beta}^\epsilon(f_1 \cdots e_i \cdots f_k) T^\beta \otimes dx_i^\vee \quad (\text{by Lemma 2.3}). \end{aligned}$$

The sign above can be simplified to $(-1)^{1-j+|f_1|+\cdots+|f_{j-1}|}$ which proves the lemma. \square

Let us make a definition abstracting the situation we are dealing with.

2.12. Definition. [Differential A_∞ algebras] A differential A_∞ algebra over a manifold \mathcal{U} is given by a triple (E, ∇, ω) such that

- E is a $\mathbb{Z}/2\mathbb{Z}$ -graded D -module over \mathcal{U} ;
- E is a sheaf of $\mathbb{Z}/2\mathbb{Z}$ -graded A_∞ algebras over \mathcal{U} ;
- ω is an even element in the de Rham complex of E .

Moreover these structures are compatible in the sense that equations 2.2 hold.

2.13. Theorem. Let (E, ∇, ω) be a differential A_∞ algebra over a smooth manifold \mathcal{U} . Then its de Rham complex $\Omega_\mathcal{U}^*(E)$ also has an A_∞ algebra structure. Explicitly its structure maps are given by (in the ϵ sign convention)

- $\hat{m}_0^\epsilon := m_0^\epsilon - \omega$;
- $\hat{m}_1^\epsilon := m_1^\epsilon + \nabla$;
- $\hat{m}_k^\epsilon := m_k^\epsilon$ for $k \geq 2$.

Proof. This is a direct computation keeping track of the signs involved. Indeed the left hand side of equation 2.2 is the additional terms resulting from adding ∇ to m_1 while the right hand side is exactly the terms we get by adding ω to the curvature term. \square

2.14. Definition. [Symplectic functions] By Lemma 2.11 the triple $(\Omega_\pi(\Lambda^\pi), \nabla, \omega)$ forms a differential A_∞ algebra over \mathcal{U} . Theorem 2.13 implies that there is an A_∞ algebra structure on the de Rham complex $\Omega_\pi(\Lambda^\pi) \otimes_{C_\mathcal{U}^\infty} \Omega_\mathcal{U}^*$. We denote this sheaf of A_∞ algebras by $\mathcal{O}_{M(\mathcal{U})}^\omega$ (or simply \mathcal{O}^ω), and refer to it as the sheaf of symplectic functions.

2.15. Deformation property of \mathcal{O}^ω . If (X, ω) is a symplectic manifold, then the triple $(C_X^\infty, d_{dR}, \omega)$ forms a differential A_∞ algebra over X whose associated curved A_∞ algebra is the de Rham algebra Ω_X^* endowed with a curvature term given by the symplectic form $-\omega$. This curved algebra may be thought of as “classical” symplectic functions, and the algebra \mathcal{O}^ω is a quantum deformation of this classical algebra.

2.16. A variant construction. In the end of this section we mention a variant of the sheaf \mathcal{O}^ω that will be used in Section 4. Namely in the above constructions we could have used the canonical model which is certain minimal model $H^*(L_\mathcal{U}, \Lambda^\pi)$ of the full de Rham complex model $\Omega^*(L_\mathcal{U}, \Lambda^\pi)$ for each Lagrangian torus fibers. All the previous constructions go through in this case as well.

More explicitly let $R\pi_*\Lambda^\pi$ be the push-forward of the constant sheaf Λ^π via the map $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$, and denote by \mathcal{H} the sheaf $R\pi_*\Lambda^\pi \otimes_{\mathbb{C}} C_\mathcal{U}^\infty$. Then \mathcal{H} has a canonical Gauss-Manin connection ∇ acting on it where we extend the action of ∇ to T^β by the same formula 2.1. Moreover the symplectic form ω can be viewed as an element in the de Rham complex of \mathcal{H} . Again locally in action-angle coordinates $\omega = \sum_{i=1}^n -dy_i \otimes dx_i^\vee = \sum_{i=1}^n -e_i \otimes dx_i^\vee$.

2.17. Theorem. *The triple $(\mathcal{H}, \nabla, \omega)$ forms a differential A_∞ algebra over \mathcal{U} . We shall denote by the resulting sheaf of A_∞ algebras by $\mathcal{O}_{M(\mathcal{U})}^{\omega, \text{can}}$ (or simply $\mathcal{O}^{\omega, \text{can}}$).*

Proof. The proof is the same as for the triple $(\Omega_\pi(\Lambda^\pi), \nabla, \omega)$. We shall not repeat it here. \square

The main advantage of $\mathcal{O}^{\omega, \text{can}}$ over \mathcal{O}^ω is that the former is of finite rank over \mathcal{U} ; while its disadvantage is that its “mirror” gives the tangent bundle of $\Omega_\mathcal{U}^*(\Lambda^\pi)$ rather than the dual torus bundle. In view of results in [21] and [14] it is likely that $\mathcal{O}^{\omega, \text{can}}$ is related to certain equivariant symplectic geometry.

3. From Lagrangians to modules

Let $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ as in the previous section, and we continue to use notations therein. In this section we construct A_∞ modules over \mathcal{O}^ω from Lagrangian branes in M . We consider Lagrangian branes of the form (L_u, α) for a Lagrangian torus fiber L_u ($u \in \mathcal{U}$) endowed with a purely imaginary torus invariant one form α on L_u . Denote by $\text{Fuk}^\pi(M)$ the full A_∞ subcategory of $\text{Fuk}(M)$ consisting of these objects. Moreover throughout this section we shall work with the following assumption.

Weak unobstructedness assumption: *For any $u \in \mathcal{U}$, the m_0 term of the curved A_∞ algebra on $\Omega(L_u, \Lambda^\pi)$ is a scalar multiple of $\mathbf{1}$, the strict unit of $\Omega(L_u, \Lambda^\pi)$.*

The main result of this section is the following theorem.

3.1. Theorem. *There exists a linear A_∞ functor $P : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}^\omega)$ which is a weak homotopy equivalence onto its image.*

Here $\text{tw}(\mathcal{O}^\omega)$ is the A_∞ category of twisted complexes over \mathcal{O}^ω possibly with internal curvatures. We refer to the Appendix A for its definition. We also note that the theorem remains true if we replace \mathcal{O}^ω by $\mathcal{O}^{\omega, \text{can}}$. This version is used in the next section to interpret mirror symmetry as Koszul duality.

3.2. Weak unobstructedness and potential function. We begin to recall the notion of weak unobstructedness from [9] Section 3.6. Consider the A_∞ algebra $\Omega(L, \Lambda^\pi)$, and denote by $\mathbf{1}$ its strict unit. An element $b \in \Omega^1(L_u, \Lambda^\pi)$ is called weak Maurer-Cartan element if we have the equation

$$\sum_{k=0}^{\infty} m_k(b^{\otimes k}) \equiv 0 \pmod{\mathbf{1}}.$$

They are important to define Lagrangian Floer homology because they give rise to deformations of the A_∞ structure on $\Omega(L_u, \Lambda^\pi)$ with square-zero differential.

In this paper we consider these elements from a more algebraic perspective: weak Maurer-Cartan elements give rise to A_∞ modules with internal curvatures. If the above equation were equal to zero (i.e. if b is an honest Maurer-Cartan element) it is well-known that such a b defines an A_∞ module over $\Omega(L_u, \Lambda^\pi)$. For the case of weak Maurer-Cartan elements we include relevant homological constructions in Appendix A. What we get in this case is an A_∞ module with an *internal curvature*. In the following we shall freely use this notion.

On the set of weak Maurer-Cartan elements we define a function called potential function by formula

$$W(\mathbf{u}, \mathbf{b}) := \text{the coefficient of } \mathbf{1} \text{ of the sum } \sum_{k=0}^{\infty} m_k(\mathbf{b}^{\otimes k}). \quad (3.1)$$

3.3. Lemma. *Let $\mathbf{b} \in H^1(L_{\mathbf{u}}, \Lambda^\pi)$ and consider it as torus invariant one-forms in $\Omega^1(L_{\mathbf{u}}, \Lambda^\pi)$. Then all such \mathbf{b} is weakly unobstructed.*

Proof. This is a direct computation using Lemma 2.3. Indeed we have

$$\begin{aligned} \sum_{k=0}^{\infty} m_k(\mathbf{b}^{\otimes k}) &= \sum_{k,\beta} m_{k,\beta}(\mathbf{b}^{\otimes k}) T^\beta \\ &= \sum_{k,\beta} \frac{1}{k!} \langle \mathbf{b}, \partial\beta \rangle^k m_{0,\beta} T^\beta \\ &= \sum_{\beta} \exp(\langle \mathbf{b}, \partial\beta \rangle) m_{0,\beta} T^\beta \\ &= \exp(\langle \mathbf{b}, \partial\beta \rangle) m_0. \end{aligned}$$

Since m_0 is by assumption a scalar multiple of $\mathbf{1}$, so is the above sum ⁷. Thus the lemma is proved. \square

3.4. Torus fibers. Let $L_{\mathbf{u}}$ be a Lagrangian torus fiber for some point $\mathbf{u} \in \mathcal{U}$. We will construct an A_∞ module $\mathcal{L}_{\mathbf{u}}$ over \mathcal{O}^ω with internal curvature $W(\mathbf{u}, 0)$. We begin to construct such a structure over the point \mathbf{u} . Since the element $\mathbf{b} = 0$ is weakly unobstructed in $\Omega(L_{\mathbf{u}}, \Lambda^\pi)$ by our assumption, it defines an A_∞ module structure on $\Omega(L_{\mathbf{u}}, \Lambda^\pi)$ over itself with internal curvature $W(\mathbf{u}, 0)$. The question is how to “propagate” this structure to other points of \mathcal{U} . For this we “propagate” the weak Maurer-Cartan element $\mathbf{b} = 0$ by a differential equation using the symplectic form ω . More precisely we define $\theta \in \mathcal{O}^\omega$ over \mathcal{U} by the following differential equation with initial condition:

$$\nabla\theta = \omega \quad \text{and} \quad \theta(\mathbf{u}) = 0.$$

Let us show that the element θ defines a weak Maurer-Cartan element of \mathcal{O}^ω with the same internal curvature $W(\mathbf{u}, 0)$ considered as a constant function over \mathcal{U} . Indeed we have (here we

⁷Note that *A priori* the sum $\sum_{k=0}^{\infty} m_k(\mathbf{b}^{\otimes k})$ only converges for $\mathbf{b} \in H^1(L_{\mathbf{u}}, \Lambda_+^\pi)$, but it follows from the explicit formula $\exp(\langle \mathbf{b}, \partial\beta \rangle)$ it also converges for $\mathbf{b} \in H^1(L_{\mathbf{u}}, \Lambda^\pi)$.

work with the ϵ sign convention)

$$\begin{aligned}
\nabla\left(\sum_{k=0}^{\infty}(-1)^{k(k-1)/2}m_k^\epsilon(\theta^k)\right) &= \sum_{k=0}^{\infty}\left[\sum_{i=1}^{k+1}(-1)^{k(k-1)/2+i-1}m_{k+1}^\epsilon(\theta^{i-1},\omega,\theta^{k-i+1})\right. \\
&\quad \left.+\sum_{j=1}^k(-1)^{k(k-1)/2+k+j-1}m_k^\epsilon(\theta^{j-1},\nabla\theta,\theta^{k-j})\right] \quad (\text{by Lemma 2.3}) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^k(-1)^{(k-1)(k-2)/2+i-1}m_k(\theta^{i-1},\omega,\theta^{k-i})- \\
&\quad -\sum_{k=1}^{\infty}\sum_{j=1}^k(-1)^{k(k-1)/2+k+j-1}m_k(\theta^{j-1},\omega,\theta^{k-j}) \quad (\text{by the equation } \nabla\theta = \omega) \\
&= 0.
\end{aligned}$$

For the last equality we observe that the sum $[(k-1)(k-2)/2+i-1]+[k(k-1)/2+k+i-1]$ from the signs is always odd, hence the two summations cancel out each other. Thus the sum $\sum_{k=0}^{\infty}(-1)^{k(k-1)/2}m_k^\epsilon(\theta^k)$ is a constant function, and is equal to $W(u,0)$ which is by definition its initial value. Apply the construction in Appendix A we get a sheaf of A_∞ modules over \mathcal{O}^ω with internal curvature $W(u,0)$.

3.5. Torus fibers with a purely imaginary closed one forms. The above construction can also be generalized to the case (L_u, α) for some $\alpha \in H^1(L_u, \mathbb{C})$ that is purely imaginary. This time we define $\theta \in \mathcal{O}^\omega$ by

$$\nabla\theta = \omega \quad \text{and} \quad \theta(u) = \alpha.$$

Lemma 3.3, together with the same computation as above, shows that θ defines a weak Maurer-Cartan element of \mathcal{O}^ω with internal curvature given by the constant function $W(u, \alpha)$ over U . We denote the associated A_∞ module by $\mathcal{L}_u(\alpha)$.

The reason we choose to work with torus invariant purely imaginary one forms in $H^1(L_u, \mathbb{C})$ is that these elements define unitary flat connections on the trivial line bundle over L_u . Note that two different one forms α_i ($i = 1, 2$) might define isomorphic unitary flat line bundle while giving rise to *different* A_∞ modules over \mathcal{O}^ω . However their endomorphism spaces $\text{End}_{\mathcal{O}^\omega}(\mathcal{L}_u(\alpha_i), \mathcal{L}_u(\alpha_i))$ ($i = 1, 2$) are canonically isomorphic. This is a direct consequence of Lemma 2.3, see for instance [11] Section 4.

3.6. Proof of Theorem 3.1. We begin to consider the endmorphism space of an object (L_u, α) in the Fukaya category. Recall by definition [9] Section 3.6 the Endomorphism complex $\text{Hom}_{\text{Fuk}(M)}((L_u, \alpha), (L_u, \alpha))$ is just the A_∞ algebra $\Omega(L_u, \Lambda^\pi)$ endowed with a differential twisted by the weak Maurer-Cartan element α . Explicitly its differential is defined as

$$m_1^\alpha(x) := \sum_{i,j=0}^{\infty} m_{i+j+1}(\alpha^i, x, \alpha^j).$$

Similarly the complex $\text{Hom}_{\mathcal{O}^\omega}(\mathcal{L}_u(\alpha), \mathcal{L}_u(\alpha))$ is the A_∞ algebra \mathcal{O}^ω twisted by the weak Maurer-Cartan element θ associated to (L_u, α) as described in the previous paragraph.

For an element $\eta \in \Omega(L_u, \Lambda^\pi)$ we define an element $P(\eta) \in \mathcal{O}^\omega$ by propagating η in a flat way. Namely $P(\eta)$ is such that $\nabla(P(\eta)) = 0$ and $P(\eta)(u) = \eta$. We shall show that the map

$$P : \text{Hom}_{\text{Fuk}(M)}((L_u, \alpha), (L_u, \alpha)) \rightarrow \text{Hom}_{\mathcal{O}^\omega}(\mathcal{L}_u(\alpha), \mathcal{L}_u(\alpha))$$

is a linear homomorphism of A_∞ algebras and a quasi-isomorphism on the underlying complexes. The fact that θ has internal curvature constant $W(u, \alpha)$ is equivalent to that P interchanges the curvature terms. This proves that P is compatible with the curvature term on both sides. To see that P is compatible with all higher multiplications m_k we need to show that

$$P[m(e^\alpha, \eta_1, e^\alpha, \dots, e^\alpha, \eta_k, e^\alpha)] = M(e^\theta, P(\eta_1), e^\theta, \dots, e^\theta, P(\eta_k), e^\theta)$$

where m and M are A_∞ structures on $\Omega(L_u, \Lambda^\pi)$ and \mathcal{O}^ω respectively, and $e^\alpha = 1 + \alpha + \alpha \otimes \alpha + \dots$ considered as an element of the bar complex of $\Omega(L_u, \Lambda^\pi)$, similarly for e^θ . The right hand side agree with the $m(e^\alpha, \eta_1, e^\alpha, \dots, e^\alpha, \eta_k, e^\alpha)$ at the point u by definition, hence it suffice to show that it is flat. Computing $\nabla[M(e^\theta, P(\eta_1), e^\theta, \dots, e^\theta, P(\eta_k), e^\theta)]$ using Lemma 2.3 and the condition $\nabla\theta = \omega$ shows that this is zero.

It remains to prove that P is a quasi-isomorphism. In fact we will show that P is a homotopy equivalence. For this we need to use the assumption that U is contractible. Let H be a homotopy between the point $u \in U$ and U . It induces a deformation retraction between functions on the point u (one dimensional) and the de Rham complex of U with coefficients in \mathbb{C} . We denote this algebraic homotopy also by H .

To extend H to $\Omega(L_u, \Lambda^\pi)$ observe that

$$P(\sum \eta_i T^{\beta_i}) = \sum P(\eta_i) P(T_i^{\beta_i}) = \sum P(\eta_i) e^{\sum_{j=1}^d \langle \partial \beta_i, e_j \rangle (x_j^\vee - u_j^\vee)} T^{\beta_i}$$

where the factor $I(\beta_i) := e^{\sum_{j=1}^d \langle \partial \beta_i, e_j \rangle (x_j^\vee - u_j^\vee)}$ is by Lemma 2.5. Thus for each sector T^β the operator $I(\beta) \circ H \circ I(\beta)^{-1}$ is a contracting homotopy, yielding a contracting homotopy between

$$(\Omega(L_u, \Lambda^\pi), 0) \simeq (\Omega_\pi(\Lambda^\pi), \nabla).$$

Here the left hand side is endowed with a zero differential while the right hand side is endowed with only ∇ as its differential.

Next we consider the operator $M_1^\theta(-) := M(e^\theta, -, e^\theta)$ on $\Omega_\pi(\Lambda^\pi)$ as a deformation of ∇ and use homological perturbation lemma to prove our theorem. For this it is enough to observe that $M_1^\theta \circ M_1^\theta = 0$ since θ is a weak Maurer-Cartan element, and that M_1^θ commutes with the homotopy H . By standard homological perturbation technique these two facts imply that H is again a homotopy between the perturbed complexes, and moreover the induced perturbed differential on $(\Omega(L_u, \Lambda^\pi))$ agrees with $m_1^\alpha(-) := m(e^\alpha, -e^\alpha)$.

The case of $\text{Hom}_{\text{Fuk}(M)}((L_u, \alpha_1), (L_u, \alpha_2))$, when we have the same Lagrangian torus fiber but endowed with different one forms, is similar. Note that by its very definition in order that this Hom space is non-vanishing it is necessary to have $W(u, \alpha_1) = W(u, \alpha_2)$, or equivalently (by 3.3) $\alpha_1 \equiv \alpha_2 \pmod{2\pi\sqrt{-1}}$. Since we work over the same point u , the propagation map P can still be defined. The previous proof carries over word-by-word to this case.

For the last case the Hom space $\text{Hom}_{\text{Fuk}(M)}((L_{u_1}, \alpha_1), (L_{u_2}, \alpha_2))$ for distinct $u_1, u_2 \in U$ is zero by definition in $\text{Fuk}(M)$. The complex $\text{Hom}_{\mathcal{O}^\omega}(\mathcal{L}_{u_1}(\alpha_1), \mathcal{L}_{u_2}(\alpha_2))$ is also zero by definition. This is due to the fact their internal curvatures are different, i.e. $W(u_1, \alpha_1) \neq W(u_2, \alpha_2)$ by Lemma 3.3. Thus the proof of Theorem 3.1 is finished. \square

Remark: The propagation equation $\nabla\theta = \omega$ is local in nature. Indeed when the base B (or the smooth part of B) has nontrivial topology, there might not exist a global solution for this equation: we do not expect to be able to write the symplectic form as an exact form.

4. Mirror symmetry and Koszul duality

In this section we study the Koszul dual algebra of the A_∞ algebra $\mathcal{O}_{M(\mathcal{U})}^{\omega, \text{can}}$ associated to a local Lagrangian torus fibration $\pi : M(\mathcal{U}) \rightarrow \mathcal{U}$ (see Section for its construction 2). We show that this Koszul dual algebra $\mathcal{O}_{\mathcal{T}\mathcal{U}}^{\text{hol}}$ is certain Dolbeault complex of the structure sheaf of the complex manifold $\mathcal{T}\mathcal{U}$ with values in the relative Novikov coefficient Λ^π (possibly with a holomorphic function as curvature). Applying a well-known correspondence of modules over Koszul dual algebras gives a natural construction of an A_∞ functor

$$\Phi^\tau : \text{tw}(\mathcal{O}^{\omega, \text{can}}) \rightarrow \text{tw}(\mathcal{O}_{\mathcal{T}\mathcal{U}}^{\text{hol}}).$$

Pre-composing with the propagation functor P defined in the previous section gives a (local) mirror functor

$$\Phi^\tau \circ P : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}_{\mathcal{T}\mathcal{U}}^{\text{hol}})$$

which is also an A_∞ functor. We prove that this local mirror functor is a quasi-equivalence onto its images. Under certain conditions we can also identify the images of this functor which turn out to be skyscraper sheaves of points in $\mathcal{T}\mathcal{U}$.

Throughout the section we continue to work with the weak unobstructedness assumption introduced in the previous section.

4.1. Potential function restricted to $\mathcal{T}\mathcal{U}$. Recall by equation 3.1 we defined a potential function on the set of weak Maurer-Cartan elements whose elements are pairs (\mathbf{u}, \mathbf{b}) for a point $\mathbf{u} \in \mathcal{U}$ and an element $\mathbf{b} \in H^1(L_{\mathbf{u}}, \Lambda^\pi)$. We consider the restriction of this function to the set of pairs (\mathbf{u}, α) where α is a purely imaginary element of $H^1(L_{\mathbf{u}}, \mathbb{C})$. The set of such pairs is canonically isomorphic to the tangent bundle $\mathcal{T}\mathcal{U}$. Locally in coordinates this identification is given by $(x_1^\vee, \dots, x_d^\vee, y_1^\vee, \dots, y_d^\vee) \mapsto (x_1^\vee, \dots, x_d^\vee, \sum_i -\sqrt{-1}y_i^\vee e_i)$.

Consider the set of Λ^π valued smooth functions on $\mathcal{T}\mathcal{U}$ which we shall denote by $C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi)$. More precisely $C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi)$ consists elements of the form $\sum_j f_j T^{\beta_j}$ for $f_j \in C_{\mathcal{T}\mathcal{U}}^\infty$, and for any energy bound E there are only finitely many nonzero terms in the series. Moreover we consider $C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi)$ as a D-module over \mathcal{U} by letting $\partial/\partial x_i^\vee$ act on it by $\bar{\partial}_i := (\partial/\partial x_i^\vee + \sqrt{-1}\partial/\partial y_i^\vee)$. Recall that the operator $\partial/\partial x_i^\vee$ acts on T^β via formula 2.1. Denote by $\Omega_{\mathcal{U}}^*(C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi)) := C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi) \otimes \Omega_{\mathcal{U}}^*$ the associated de Rham complex with coefficients in $C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi)$. The differential on this de Rham complex will be denoted by $\bar{\partial}$.

The notation $\bar{\partial}$ has a geometric explanation. If we assume all convergence in Lagrangian Floer theory, we can evaluate T at e^{-1} and work over \mathbb{C} . Then $\Omega^*(C_{\mathcal{T}\mathcal{U}}^\infty(\Lambda^\pi))$ after evaluating T at e^{-1} can be identified with the classical Dolbeault differential $\bar{\partial}$ of the structure sheaf of $\mathcal{T}\mathcal{U}$.

4.2. Lemma. *We have $\bar{\partial}W = 0$. In the geometric situation this implies W is a holomorphic function on $\mathcal{T}\mathcal{U}$.*

Proof. By the identification $(y_1^\vee, \dots, y_d^\vee) \mapsto \sum_{i=1}^d -\sqrt{-1}y_i^\vee e_i$ mentioned above, we evaluate W at $b = \sum_{i=1}^d -\sqrt{-1}y_i^\vee e_i$ and differentiate. We have

$$\begin{aligned} \sqrt{-1}\partial_{y_i^\vee}(m_{k+1}(b, \dots, b)) &= \sum_{j=1}^{k+1} m_{k+1}(b, \dots, e_i, \dots, b) \quad (e_i \text{ in the } j\text{-th spot}) \\ &= -\partial_{x_i^\vee} m_k(b, \dots, b) \quad (\text{by Lemma 2.11}). \end{aligned}$$

Summing over k yields the result. \square

4.3. Koszul dual of $\mathcal{O}_{M(\mathcal{U})}^{\omega, \text{can}}$. The Koszul dual algebra $\mathcal{O}_{\mathcal{U}}^{\text{hol}}$ (or simply \mathcal{O}^{hol}) of $\mathcal{O}^{\omega, \text{can}}$ is defined as follows. This is a sheaf of curved differential graded algebras over \mathcal{U} . Its underlying differential graded algebra is simply $\Omega_{\mathcal{U}}^*(C_{\mathcal{U}}^\infty(\Lambda^\pi)) = C_{\mathcal{U}}^\infty(\Lambda^\pi) \otimes \Omega_{\mathcal{U}}^*$ endowed with the differential $\bar{\partial}$ and the natural tensor product algebra structure. Its curvature term is $-W$ (the sign is due to Theorem 4.4 below). By the lemma above this defines a curved differential graded algebra. Geometrically we think of it as the Dolbeault complex of \mathcal{U} endowed with a curvature term $-W$.

The two sheaves of algebras $\mathcal{O}^{\omega, \text{can}}$ and \mathcal{O}^{hol} are Koszul dual to each other in the sense of the following theorem.

4.4. Theorem. *Define the element $\tau := \sum_{i=1}^d -\sqrt{-1}y_i \otimes e_i$ in an affine coordinates of \mathcal{U} (note that since e_i and y_i are dual to each other, the element τ is independent of coordinates). Then τ is a MC element of the tensor product A_∞ algebra $\mathcal{O}^{\text{hol}} \otimes_{\Omega_{\mathcal{U}}^*(\Lambda^\pi)} \mathcal{O}^{\omega, \text{can}}$.*

Proof. This is a straight forward computation except that we need to pay extra attention to the signs. Indeed to form the tensor product A_∞ algebra we use the ϵ sign convention as explained in more detail in the appendix. Let us denote by M_k^ϵ the resulting structure constants. Then we have

$$\begin{aligned} M_0^\epsilon &= m_0^\epsilon - \omega - W; \\ M_1^\epsilon(\tau) &= m_1^\epsilon(\tau) + \nabla(\tau) + \bar{\partial}(\tau); \\ M_k^\epsilon(\tau) &= m_k^\epsilon(\tau, \dots, \tau) \quad \text{for } k \geq 2. \end{aligned}$$

Observe that the sum $\sum_{k=0}^\infty (-1)^{k(k-1)/2} m_k(\tau, \dots, \tau)$ is by definition W . Moreover we have $\nabla(\tau) = 0$ and $\bar{\partial}(\tau) = \sum dx_i \otimes e_i = \omega$. Thus summing over these equalities we get

$$\sum_{k=0}^\infty (-1)^{k(k-1)/2} M_k^\epsilon(\tau, \dots, \tau) = 0.$$

Thus the theorem is proved. \square

4.5. Local construction of mirror functor. Let A be an A_∞ algebra and B be a curved differential graded algebra over a base ring R . Assume both A and B are free R -modules, and that A is of finite rank of R . Then, as is explained in the Appendix B, associated to any Maurer-Cartan element $\tau \in B \otimes A$ we get an A_∞ functor $\Phi^\tau : \text{tw}(A) \rightarrow \text{tw}(B)$. Intuitively speaking the Maurer-Cartan element τ to twist the tensor product $B \otimes A$ to get a rank one twisted complex over $B \otimes A$. Viewing this $B \otimes A$ -module as a kernel we get a functor from $\text{tw}(A)$ to $\text{tw}(B)$. This construction is straightforward for ordinary algebras A and B , but requires more

explanations in the A_∞ setting. Moreover we also would like to include modules with internal curvatures into this construction.

Let us apply this construction to the case $A = \mathcal{O}^{\omega, \text{can}}$ and $B = \mathcal{O}_{\mathbb{U}}^{\text{hol}}$ over the base ring $R = \Omega_{\mathbb{U}}^*(\Lambda^\pi)$, which yields an A_∞ functor

$$\Phi^\tau : \text{tw}(\mathcal{O}^{\omega, \text{can}}) \rightarrow \text{tw}(\mathcal{O}^{\text{hol}}).$$

Next we explicitly describe this local mirror functor Φ^τ . For an A_∞ module \mathcal{L} over $\mathcal{O}^{\omega, \text{can}}$ (in particular \mathcal{L} must be a $\Omega_{\mathbb{U}}^*(\Lambda^\pi)$ -module since this is our base ring), define an \mathcal{O}^{hol} -module $\Phi^\tau(\mathcal{L})$ as follows. As a sheaf over \mathbb{U} this is just $\mathcal{O}_{\mathbb{U}}^{\text{hol}} \otimes_{\Omega_{\mathbb{U}}^*(\Lambda^\pi)} \mathcal{L}$. Its $\mathcal{O}_{\mathbb{U}}^{\text{hol}}$ -module structure is induced from the first tensor component. On the tensor product we put a twisted differential defined by formula

$$d := \bar{\partial} \otimes \text{id} + \sum_{k=0}^{\infty} \hat{\rho}_k(\tau, \dots, \tau).$$

If we denote by $\rho_k : (\mathcal{O}^{\omega, \text{can}})^{\otimes k} \otimes_{\Omega_{\mathbb{U}}^*(\Lambda^\pi)} \mathcal{L} \rightarrow \mathcal{L}$ the structure constants of the A_∞ module \mathcal{L} , then the maps $\hat{\rho}_k(\tau, \dots, \tau) : \mathcal{O}^{\text{hol}} \otimes_{\Omega_{\mathbb{U}}^*(\Lambda^\pi)} \mathcal{L} \rightarrow \mathcal{O}^{\text{hol}} \otimes_{\Omega_{\mathbb{U}}^*(\Lambda^\pi)} \mathcal{L}$ is defined by

$$\hat{\rho}_k(\tau, \dots, \tau)(f \otimes m) := \sum_{i_1, i_2, \dots, i_k} y_{i_1} \cdots y_{i_k} \cdot f \otimes \rho_k(e_{i_1}, \dots, e_{i_k}; m)$$

up to signs. The A_∞ functors on Hom spaces can also be described explicitly. We refer to Appendix B for more details.

4.6. Mirror of torus fibers. Let us identify the object $\Phi^\tau(\mathcal{L}_u(\alpha))$ for the \mathcal{O}^ω -module \mathcal{L}_u associated to a torus fiber L_u endowed with a purely imaginary one form $\alpha \in H^1(L_u, \mathbb{C})$ on it. We refer to the previous section 3 for the construction of $\mathcal{L}_u(\alpha)$.

4.7. Definition. Let u be a point in \mathbb{U} , and let α be a purely imaginary one form in $H^1(L_u, \mathbb{C})$. Then the $\mathcal{O}_{\mathbb{U}}^{\text{hol}}$ -module $\Lambda^\pi(u, \alpha)$ is defined as follows. As a sheaf over \mathbb{U} it is simply the skyscraper sheaf Λ^π over the point $u \in \mathbb{U}$. The $\mathcal{O}_{\mathbb{U}}^{\text{hol}}$ module structure is defined by letting an element $f \in \mathcal{O}^{\text{hol}}$ acting on Λ^π via multiplication by $f(u, \alpha)$.

4.8. Proposition. Assume that strictly negative Maslov index does not contribute to structure maps m_k , and assume further that the potential function $W = 0$. Then the object $\Phi^\tau(\mathcal{L}_u(\alpha))$ is quasi-isomorphic to $\Lambda^\pi(u, \alpha)[d]$.

Proof. Let e_1, \dots, e_d be a trivialization of $R^1\pi_*\mathbb{Z}$ which induces a trivialization of $R\pi_*\mathbb{Z}$ whose flat sections are $e_I := e_{i_1} \wedge \dots \wedge e_{i_j}$ for any subset $I \subset \{1, \dots, d\}$. This trivialization defines an isomorphism of sheaves on \mathbb{U}

$$\Phi^\tau(\mathcal{L}_u(\alpha)) \cong \prod_{I \subset \{1, \dots, d\}} \mathcal{O}_{\mathbb{U}}^{\text{hol}} \otimes e_I.$$

Using this trivialization we can explicitly write down the differential on $\Phi^\tau(\mathcal{L}_u(\alpha))$. The “untwisted differential” on this sheaf is simply $\bar{\partial}$. By Theorem B.4 in Appendix A the twisted part is given by

$$Q(f \otimes e_I) := \sum_{k \geq 0, l \geq 0} m_{k+l+1}(\tau^l, f \otimes e_I, \theta^k).$$

Here τ is as in Theorem 4.4, and recall that θ was defined by differential equation $\nabla\theta = \omega$ with initial condition $\theta(u) = \alpha$. Again by Theorem B.4 we have

$$(W(u, \alpha) - W) \text{id} + [\bar{\partial}, Q] - Q^2 = 0.$$

Since W is assumed to be zero this equation simplifies to $[\bar{\partial}, Q] - Q^2 = 0$. Moreover by degree reason we have $[\bar{\partial}, Q] = 0$ and $Q^2 = 0$. The equation $[\bar{\partial}, Q] = 0$ implies that Q is a holomorphic operator.

To understand the cohomology of the differential $\bar{\partial} + Q$ on $\Phi^\tau(\mathcal{L}_u(\alpha))$, we first kill the $\bar{\partial}$ part. For this define a morphism

$$F_1 : (\mathcal{A}^\pi \otimes \prod_{I \subset \{1, \dots, d\}} e_I, Q) \rightarrow \Phi^\tau(\mathcal{L}_u(\alpha)) = (\mathcal{O}_{\mathbb{T}u}^{\text{hol}} \otimes \prod_{I \subset \{1, \dots, d\}} e_I, \bar{\partial} + Q)$$

where \mathcal{A}^π denotes kernel of the operator $\bar{\partial} : C_{\mathbb{T}u}^\infty(\Lambda^\pi) \rightarrow C_{\mathbb{T}u}^\infty(\Lambda^\pi) \otimes \Omega_u^1$, and the map F_1 is simply the embedding map $\mathcal{A}^\pi \otimes e_I \hookrightarrow \mathcal{O}_{\mathbb{T}u}^{\text{hol}} \otimes e_I$. The fact that F_1 is a quasi-isomorphism follows from the acyclicity of Dolbeault complex. We can also explicitly describe \mathcal{A}^π . This is the set of formal sums of the form

$$\sum_{\beta} e^{\sum_i \langle \partial\beta, e_i \rangle x_i^\vee} \cdot f_\beta \cdot T^\beta$$

for holomorphic functions f_β on $\mathbb{T}u$. These formal series satisfies the same finiteness condition as in the definition of Λ^π . Note that here the extra term $e^{\sum_i \langle \partial\beta, e_i \rangle x_i^\vee}$ appears due to the non-trivial action of $\bar{\partial}$ on T^β by formula 2.1.

The cohomology of $(\mathcal{A}^\pi \otimes \prod_{I \subset \{1, \dots, d\}} e_I, Q)$ can be related to $\Lambda^\pi(u, \alpha)[d]$ by defining another map

$$F_2 : (\mathcal{A}^\pi \otimes \prod_{I \subset \{1, \dots, d\}} e_I, Q) \rightarrow \Lambda^\pi(u, \alpha)[d]$$

which maps $f \otimes e_I$ to zero unless $I = \{1, \dots, d\}$ in which case we define $F_2(f \otimes e_1 \wedge \dots \wedge e_d) := f(u, \alpha)$. It is clear from the definition that F_2 is compatible with the $\mathcal{O}_{\mathbb{T}u}^{\text{hol}}$ -modules structures.

4.9. Lemma. *The $\mathcal{O}_{\mathbb{T}u}^{\text{hol}}$ -morphism F_2 is a map of complexes.*

Proof of Lemma. Below we refer to the wedge degree of e_I the integer $|I|$. Recall the structure maps $m_{k,\beta}$ are of wedge degree $2 - k - \mu(\beta)$. Thus the operator Q_β

$$e_I \mapsto \sum_{k \geq 0, l \geq 0} m_{k+l+1, \beta}(\tau^l, e_I, \theta^k)$$

is of wedge degree $1 - \mu(\beta)$. This is because locally $\tau = \sum_{i=1}^d -\sqrt{-1}y_i^\vee e_i$ and $\theta = \sum_{i=1}^d (x_i^\vee - u_i - \sqrt{-1}\alpha_i)e_i$ which are both of wedge degree one. By our assumption negative Maslov index disks do not contribute to m_k , it follows that Q_β can only increase the wedge degree when $\beta = 0$ corresponding to constant maps. By constructions in [7] this zero energy part is precisely the exterior algebra on cohomology of torus⁸. By formula of τ and θ we get this zero energy part is

$$Q_0(e_I) = m_{2,0}(\tau, e_I) + m_{2,0}(e_I, \theta) = \sum_i (x_i^\vee + \sqrt{-1}y_i^\vee - u_i - \sqrt{-1}\alpha_i)e_i \wedge e_I$$

⁸It is essential for this proof that the zero energy part of $H^*(L_u, \Lambda^\pi)$ is given by the exterior product as opposed to an A_∞ algebra homotopy equivalent to it. The latter was proved for constructions in [9] Section 7.5.

which vanishes at $x = u$ and $y = \alpha$. Thus we have shown that F_2 is a map of complexes. \square

To prove the proposition it remains to show that F is a quasi-isomorphism. For this we make use of the spectral sequence associated the energy filtration on both sides which is valid since F preserves this filtration (we refer to [9] Chapter 6 for details of the construction of this spectral sequence). We claim that the first page of this spectral sequence is already an isomorphism. The first page is obtained as the cohomology of the operator Q_0 . Since the factor $e^{\sum_i \langle \partial \beta, e_i \rangle x_i^\vee}$ is non-vanishing, the cohomology for each $\beta \in G$ is the same. Thus it suffices to analyze the case when $\beta = 0$. That is we would like to show that the map

$$F_0 : \left(\prod_{I \subset \{1, \dots, d\}} \mathcal{A}(\mathbb{T}u) \otimes e_I, Q_0 \right) \rightarrow \mathbb{C}(u, \alpha)[d]$$

defined by evaluating at (u, α) is a quasi-isomorphism where $\mathcal{A}(\mathbb{T}u)$ is the sheaf of holomorphic functions on $\mathbb{T}u$. To this end we observe that Q_0 acts on the cohomology of $\bar{\partial}$ as the classical Koszul differential. It is well-known that the cohomology of this Koszul complex is $\mathbb{C}(u, \alpha)$ concentrated in degree d . Note that it is important here that we deal with holomorphic functions since the acyclicity of Koszul complex fails for C^∞ functions. The proposition is proved. \square

Remark: If M is a compact symplectic manifold with vanishing first Chern class (computed with a choice of almost complex structure), and L_u a *special* Lagrangian submanifold, then the assumptions in the above proposition are satisfied. In fact in this case all the map $\mu : G(L_u) \rightarrow 2\mathbb{Z}$ is identically zero.

Remark: The assumption on Maslov index is more of a technical nature while the condition that $W = 0$ is necessary to describe the object $\Phi^\tau(\mathcal{L}_u(\alpha))$ as a skyscraper sheaf since otherwise $\Phi^\tau(\mathcal{L}_u(\alpha))$ is only a matrix factorization of $W - W(u, \alpha)$ which is not a complex itself. In general without these assumptions we have the following theorem.

4.10. Theorem. *The composition of A_∞ functors*

$$\Phi^\tau \circ P : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}_{\mathbb{T}u}^{\text{hol}})$$

is a homotopy equivalence onto its image.

Proof. By Theorem 3.1 the first functor is in fact a linear A_∞ functor which is a homotopy equivalence onto its image. Thus it remains to show this for the second functor Φ^τ . That is we would like to show that

$$\Phi^\tau : \text{Hom}_{\mathcal{O}^{\omega, \text{can}}}(\mathcal{L}_{u_1}(\alpha_1), \mathcal{L}_{u_2}(\alpha_2)) \rightarrow \text{Hom}_{\mathcal{O}^{\text{hol}}}(\Phi^\tau(\mathcal{L}_{u_1}(\alpha_1)), \Phi^\tau(\mathcal{L}_{u_2}(\alpha_2)))$$

is a quasi-isomorphism. For this we can argue in the way as in the proof of Proposition 4.8.

In the case when $u_1 \neq u_2$ or $\alpha_1 \neq \alpha_2$ it was shown in the previous section that the complex $\text{Hom}_{\mathcal{O}^{\omega, \text{can}}}(\mathcal{L}_{u_1}(\alpha_1), \mathcal{L}_{u_2}(\alpha_2))$ has zero cohomology. On the other hand the complex $\text{Hom}_{\mathcal{O}^{\text{hol}}}(\Phi^\tau(\mathcal{L}_{u_1}(\alpha_1)), \Phi^\tau(\mathcal{L}_{u_2}(\alpha_2)))$ also has vanishing cohomology. For this we can use the spectral sequence associated to the energy filtration to calculate its cohomology. The first page of this spectral sequence is already zero due to the vanishing of $\text{Ext}_{\mathbb{T}u}^*(\mathcal{O}_{(u_1, \alpha_1)}, \mathcal{O}_{(u_2, \alpha_2)})$.

Thus in the following we consider the case when $u_1 = u_2 = u$ and $\alpha_1 = \alpha_2 = \alpha$. We can kill the Ω_u^* part by observing the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fuk}^\pi(\mathcal{M})}((L_u, \alpha), (L_u, \alpha)) & \xrightarrow{\varphi^\tau} & \mathcal{A}^\pi \otimes \mathrm{End}_{\mathbb{C}}(H^*(L_u, \mathbb{C})) \\ \mathrm{p} \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}^{\omega, \mathrm{can}}}(\mathcal{L}_u(\alpha), \mathcal{L}_u(\alpha)) & \xrightarrow{\Phi^\tau} & \mathrm{Hom}_{\mathcal{O}^{\mathrm{hol}}}(\Phi^\tau(\mathcal{L}_u(\alpha)), \Phi^\tau(\mathcal{L}_u(\alpha))) \end{array}$$

Several explanations of this diagram are in order. First of all as in the proof of Proposition 4.8, \mathcal{A}^π is holomorphic functions on $\mathbb{T}u$ with values in Λ^π , and the right vertical map is the inclusion

$$\mathcal{A}^\pi \otimes \mathrm{End}_{\mathbb{C}}(H^*(L_u, \mathbb{C})) \hookrightarrow \mathcal{O}^{\mathrm{hol}} \otimes \mathrm{End}_{\mathbb{C}}(H^*(L_u, \mathbb{C}))$$

which is a quasi-isomorphism proved in Proposition 4.8. Moreover by definition we have $\mathcal{O}^{\mathrm{hol}} \otimes \mathrm{End}_{\mathbb{C}}(H^*(L_u, \mathbb{C})) \cong \mathrm{Hom}_{\mathcal{O}^{\mathrm{hol}}}(\Phi^\tau(\mathcal{L}_u(\alpha)), \Phi^\tau(\mathcal{L}_u(\alpha)))$, which implies that the right vertical arrow is also a quasi-isomorphism.

Secondly the left vertical arrow is the propagation map defined in Theorem 3.1 which is also a quasi-isomorphism. Thus in order to show the bottom arrow Φ^τ is a quasi-isomorphism it suffices to define the top arrow φ^τ and that it is a quasi-isomorphism.

Explicitly the map φ^τ is defined by

$$\varphi^\tau(e_I) := 1 \otimes e_I^\vee \otimes \sum_{l \geq 0, i_0 \geq 0, i_1 \geq 0} m_{l+2+i_0+i_1}(\tau^l, e_J, \theta^{i_0}, e_I, \theta^{i_1})$$

where as before $\tau = \sum -\sqrt{-1}y_i^\vee e_i$ and $\theta = \sum (x_i^\vee - u_i - \sqrt{-1}\alpha)e_i$. This formula is the same as the formula to define Φ^τ , and hence the above diagram is commutative, see Appendix B.

It remains to show that φ^τ is a quasi-isomorphism. For this we consider the spectral sequences associated energy filtrations on both sides. The first page of this spectral sequence is already an isomorphism, again it suffices to prove that the zero energy part sector (corresponding to $\beta = 0$) is an isomorphism, which reduces to show that the map of complexes

$$\begin{aligned} (H^*(L_u, \mathbb{C}), 0) &\rightarrow (\mathrm{End}_{\mathcal{A}(\mathbb{T}u)}(\mathcal{A}(\mathbb{T}u) \otimes H^*(L_u, \mathbb{C})), [Q_0, -]) \\ e_I &\mapsto (f \otimes e_J \mapsto f \otimes e_I \wedge e_J) \end{aligned}$$

where $\mathcal{A}(\mathbb{T}u)$ is holomorphic functions on $\mathbb{T}u$ and Q_0 is the Koszul differential associated to the regular sequence $\{x_i^\vee + \sqrt{-1}y_i^\vee - u_i - \sqrt{-1}\alpha\}_{i=1}^d$, is a quasi-isomorphism. This is an exercise in classical Koszul duality theory between exterior algebras and symmetric algebras. \square

Remark: Without the assumptions in Proposition 4.8 it is not clear how to identify the image of Φ^τ in $\mathrm{tw}(\mathcal{O}^{\mathrm{hol}})$. In general these objects are matrix factorizations of W minus a constant (the internal curvatures), and we expect them to be homotopy equivalent to stabilization matrix factorizations introduced in [6] Section 2.3. The trouble to prove this claim is that the matrix factorization defined by the operator $Q(f \otimes e_I) := \sum_{k \geq 0, l \geq 0} m_{k+l+1}(\tau^l, f \otimes e_I, \theta^k)$ mixes various the wedge degrees of e_I while the stabilizations introduced in [6] only involves operators of wedge degrees 1 and -1 .

5. Mirror symmetry and Fourier-Mukai transform

From the symplectic point of view the two sheaves of A_∞ algebras $\mathcal{O}^{\omega, \text{can}}$ and \mathcal{O}^ω are not much different since $\mathcal{O}^{\omega, \text{can}}$ in some sense is the minimal model of \mathcal{O}^ω . However from the point of view of mirror symmetry there is an important difference between the two. Indeed we have seen from the previous section that the mirror of $\mathcal{O}^{\omega, \text{can}}$ is $\mathbb{T}\mathbb{U}$ while in this section we show the mirror of \mathcal{O}^ω is the dual torus bundle $M^\vee(\mathbb{U})$. Analogous results such as Proposition 4.8 and Theorem 4.10 are obtained in this case as well.

Throughout the section we continue to work with the unobstructedness assumption introduced in the previous section.

5.1. The case without quantum corrections. Let us first consider the case when there exists no non-trivial holomorphic disks in M with boundary in L_u for all $u \in \mathbb{U}$. In this case we can work over \mathbb{C} . We shall see how relative (over \mathbb{U}) Poincaré bundles appear here.

In Theorem 4.4 we constructed a Maurer-Cartan element in the tensor product algebra $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^{\omega, \text{can}}$ which is of the form $\tau := \sum_{i=1}^d -\sqrt{-1}y_i^\vee \otimes e_i$. Considering elements e_i as translation invariant one forms in $\Omega^1(L_u)$ the element τ can be viewed as an element in $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$. As is explained in Appendix B Koszul duality can be considered as a special case of Fourier-Mukai transform. Indeed the kernel to construct the Koszul functor $\Phi^\tau : \text{tw}(\mathcal{O}^\omega) \rightarrow \text{Tw}(\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}})$ ⁹ is simply the rank one twisted complex over $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$ defined by the Maurer-Cartan element τ .

We can explicitly describe this twisted complex in coordinates x^\vee , y and y^\vee . Recall x^\vee is coordinates on the base \mathbb{U} ; y and y^\vee are periodic coordinates on Lagrangian torus and dual torus. We also trivialize $M(\mathbb{U})$ to $\mathbb{T} \times \mathbb{U}$, and denote by $H_1(\mathbb{T}, \mathbb{R})$ by V , its dual by V^\vee . Then the tensor product $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$ is simply $C_{V^\vee}^\infty \otimes \Omega^*(\mathbb{T}) \otimes \Omega_\mathbb{U}^*$ ¹⁰ endowed with the differential $d_\mathbb{T} + \bar{\partial}$. We recall that the operator $\bar{\partial}$ is defined as $\sum_{i=1}^d (\partial/\partial x_i^\vee + \sqrt{-1}\partial/\partial y_i^\vee) dx_i^\vee$. Using the Maurer-Cartan element τ this differential is twisted to $d_\mathbb{T} - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}$. Note that the square of this twisted operator is not zero but the symplectic form ω . Indeed this should be a twisted complex over $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$ which has curvature $-\omega$. We denote this twisted complex by K^τ .

To pass from $\mathbb{T}\mathbb{U} = V^\vee \times \mathbb{U}$ to $M^\vee(\mathbb{U}) = \mathbb{T}^\vee \times \mathbb{U}$ it suffices to quotient out the dual lattice group $\Gamma := (H_1(\mathbb{T}, \mathbb{Z}))^\vee = H_1(\mathbb{T}^\vee, \mathbb{Z})$ in V^\vee . However we note that the kernel does not descend to this quotient in an obvious way since the twisted operator $d_\mathbb{T} - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}$ is not Γ -equivariant under the natural translation action. This is where Poincaré bundle comes into play: we can define another Γ action on K^τ so that the operator $d_\mathbb{T} - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}$ becomes equivariant. This “twisted action” is given by

$$\gamma[f(y, y^\vee)] := e^{\sqrt{-1}\gamma \cdot y} f(y, y^\vee - \gamma) \quad (5.1)$$

where $\gamma \cdot y$ is the natural pairing between V^\vee and V . It is well-known that if we take the above action and consider invariants in the function part $C_{V^\vee}^\infty \otimes C_\mathbb{T}^\infty \otimes C_\mathbb{U}^\infty$ of K^τ we get C^∞ -sections of the relative Poincaré bundle \mathcal{P} on $M(\mathbb{U}) \times_\mathbb{U} M^\vee(\mathbb{U})$.

⁹Here we need to use the capital Tw since \mathcal{O}^ω is of infinite rank over $\Omega_\mathbb{U}^*(\Lambda)$.

¹⁰Here and in the following \otimes means completed tensor product.

A direct computation verifies the following commutative diagram

$$\begin{array}{ccc} K^\tau & \xrightarrow{d_T - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}} & K^\tau \\ \gamma \downarrow & & \downarrow \gamma \\ K^\tau & \xrightarrow{d_T - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}} & K^\tau. \end{array}$$

Thus the operator $d_T - \sqrt{-1}y_i^\vee dy_i + \bar{\partial}$ descends to an operator on invariants $(K^\tau)^\Gamma$. Moreover the $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ -module structure

$$\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes K^\tau \rightarrow K^\tau$$

is Γ -equivariant if we put the ordinary translation action on $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ and the twisted action on K^τ . Taking invariants yields an action

$$\mathcal{O}_{M^\vee(\mathbb{U})}^{\text{hol}} \otimes (K^\tau)^\Gamma \rightarrow (K^\tau)^\Gamma.$$

where $\mathcal{O}_{M^\vee(\mathbb{U})}^{\text{hol}}$ is the Γ -invariants of $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ under translation action, i.e. it is the Dolbeault complex of the structure sheaf of the complex manifold $M^\vee(\mathbb{U})$.

We can then consider $(K^\tau)^\Gamma$ as a “twisted complex” over $\mathcal{O}_{M^\vee(\mathbb{U})}^{\text{hol}} \otimes \mathcal{O}^\omega$. Here we put twisted complex in quote since the two objects $(K^\tau)^\Gamma$ and $\mathcal{O}_{M^\vee(\mathbb{U})}^{\text{hol}} \otimes \mathcal{O}^\omega$ are not isomorphic even as sheaves on \mathbb{U} ¹¹. As in the previous section we would like to use $(K^\tau)^\Gamma$ as a Kernel to define a local mirror symmetry functor. We shall perform this construction in the general case when quantum corrections are presented.

5.2. Quantum Fourier-Mukai transform. Let us return to the general case to allow non-trivial holomorphic disks to enter the picture. The previous discussion motivates us to perform the construction in two steps:

- I. Replace the sheaf $\mathcal{O}^{\omega, \text{can}}$ by \mathcal{O}^ω , and perform the same construction as in the previous section;
- II. Descend to Γ -invariants by action 5.1.

We begin with Step I which is almost word by word as in the previous section. Consider the \mathcal{O}^ω -module $\mathcal{L}_u(\alpha)$ associated to a Lagrangian L_u endowed with a purely imaginary closed one form α (see Section 3 for its definition). This is a twisted complex of rank one over \mathcal{O}^ω defined by a Maurer-Cartan element θ such that $\nabla\theta = \omega$ and $\theta(u) = \alpha$. In local coordinates $\theta = \sum_i (x_i^\vee - u_i + \alpha)e_i$, and let $\tau := \sum_i -\sqrt{-1}y_i^\vee e_i \in \mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$ be as before. By constructions in the previous section we get a twisted complex $\Phi^\tau(\mathcal{L}_u(\alpha))$ over $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ which, as a $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}}$ -module, is simply $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$. It is endowed with a twisted differential $d := \bar{\partial} + Q$ where the operator Q is

$$Q(?) = \sum_{k \geq 0, l \geq 0} m_{k+l+1}(\tau^l, ?, \theta^k)$$

expressed using structure maps on the tensor product A_∞ algebra $\mathcal{O}_{\mathbb{T}\mathbb{U}}^{\text{hol}} \otimes \mathcal{O}^\omega$.

For the step II we first observe that it follows from Lemma 2.3 the potential function W , *A priori* defined on $\mathbb{T}\mathbb{U}$, is in fact a function on $M^\vee(\mathbb{U})$ with values in Λ^π . This enables us

¹¹Indeed the Poincaré bundle \mathcal{P} is not a topologically trivial vector bundle.

to define the sheaf of curved algebras $\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$ in general. Our next goal is to show that the operator $d := \bar{\partial} + Q$ on $\mathcal{O}_{\text{Tu}}^{\text{hol}} \otimes \mathcal{O}^\omega$ intertwines with action 5.1. This follows from the following lemma.

5.3. Lemma. *For each $\beta \in G$, and $f \in \mathcal{O}_{\text{Tu}}^{\text{hol}} \otimes \mathcal{O}^\omega$ we have*

$$\sum_{l \geq 0} m_{l+1+k, \beta}(\tau^l, \gamma(f), \theta^k) = \gamma \left[\sum_{l \geq 0} m_{l+1+k, \beta}(\tau^l, f, \theta^k) \right].$$

Proof. We denote by T_γ the translation action $\mathbf{y}^\vee \mapsto \mathbf{y}^\vee - \gamma$. Thus $\gamma(f) = e^{\sqrt{-1}\gamma \cdot \mathbf{y}}$. We have

$$\begin{aligned} m_{l+1+k, \beta}(\tau^l, \gamma(f), \theta^k) &= (\text{ev}_0)! \left[\int_{0 \leq t_1 \leq t_2 \leq \dots \leq t_{k+l+1} \leq 1} \text{ev}^*(\tau^l) \cdot \text{ev}_{l+1}^*(\gamma(f)) \cdot \text{ev}^*(\theta^k) \right] \\ &= (\text{ev}_0)! \left[\frac{1}{l!} \langle \partial\beta, \tau \rangle^l \int_{0 \leq t_1 \leq \dots \leq t_{k+l+1} \leq 1} t_{l+1}^l \text{ev}_{l+1}^*(\gamma(f)) \cdot \text{ev}^*(\theta^k) \right] \\ &= (\text{ev}_0)! \left[\frac{1}{l!} \langle \partial\beta, \tau \rangle^l \int_{0 \leq t_1 \leq \dots \leq t_{k+l+1} \leq 1} t_1^l \text{ev}_1^*(e^{\sqrt{-1}\gamma \cdot \mathbf{y}}) \text{ev}_1^*(T_\gamma f) \cdot \text{ev}^*(\theta^k) \right] \\ &= (\text{ev}_0)! \left[\text{ev}_0^*(e^{\sqrt{-1}\gamma \cdot \mathbf{y}}) \frac{\langle \partial\beta, \tau \rangle^l}{l!} \int_{0 \leq t_1 \leq \dots \leq t_{k+l+1} \leq 1} t_1^l e^{\sqrt{-1}\gamma \cdot (t_1 \partial\beta)} \text{ev}_1^*(T_\gamma f) \cdot \text{ev}^*(\theta^k) \right]. \end{aligned}$$

The last equality follows from the map $\text{ev}_1 : \mathcal{M}_{1+1, \beta} \cong \mathcal{M}_{0+1, \beta} \times [0, 1] \rightarrow \mathbf{L}$ is $\text{ev}_1([u, t]) = \text{ev}_0([u]) + t \cdot \partial\beta$. Apply projection formula $f_!(f^*a \cdot b) = a \cdot f_!b$ to the last summation yields

$$e^{\sqrt{-1}\gamma \cdot \mathbf{y}} (\text{ev}_0)! \left[\frac{1}{l!} \langle \partial\beta, \tau \rangle^l \int_{0 \leq t_1 \leq \dots \leq t_{k+l+1} \leq 1} t_1^l e^{\sqrt{-1}\gamma \cdot (t_1 \partial\beta)} \text{ev}_1^*(T_\gamma f) \cdot \text{ev}^*(\theta^k) \right].$$

Summing over l yields

$$\begin{aligned} \sum_{l \geq 0} m_{l+1+k, \beta}(\tau^l, \gamma(f), \theta^k) &= e^{\sqrt{-1}\gamma \cdot \mathbf{y}} (\text{ev}_0)! \left[\int_{0 \leq t_1 \leq \dots \leq t_{k+1} \leq 1} e^{(t_1 \partial\beta) \cdot \tau} e^{\sqrt{-1}\gamma \cdot (t_1 \partial\beta)} \text{ev}_1^*(T_\gamma f) \cdot \text{ev}^*(\theta^k) \right] \\ &= e^{\sqrt{-1}\gamma \cdot \mathbf{y}} (\text{ev}_0)! \left[\int_{0 \leq t_1 \leq \dots \leq t_{k+1} \leq 1} e^{(t_1 \partial\beta) \cdot T_\gamma \tau} \text{ev}_1^*(T_\gamma f) \cdot \text{ev}^*(\theta^k) \right] \\ &= \gamma \left\{ (\text{ev}_0)! \left[\int_{0 \leq t_1 \leq \dots \leq t_{k+1} \leq 1} e^{(t_1 \partial\beta) \cdot \tau} \text{ev}_1^*(f) \cdot \text{ev}^*(\theta^k) \right] \right\} \\ &= \gamma \left\{ \sum_{l \geq 0} (\text{ev}_0)! \left[\int_{0 \leq t_1 \leq \dots \leq t_{k+1} \leq 1} \frac{1}{l!} \langle \partial\beta, \tau \rangle^l t_1^l \cdot \text{ev}_1^*(f) \cdot \text{ev}^*(\theta^k) \right] \right\} \\ &= \gamma \left[\sum_{l \geq 0} m_{l+1+k, \beta}(\tau^l, f, \theta^k) \right]. \end{aligned}$$

Thus the lemma is proved. \square

Remark: It follows from the proof that in the above formula the part θ^k can be replaced by any elements.

Returning to the discussion of the twisted complex $\Phi^\tau(\mathcal{L}_u(\alpha))$ endowed with differential $\bar{\partial} + Q$, the above lemma implies that the operator Q is Γ -equivariant. Since $\bar{\partial}$ is also equivariant the sum operator $\bar{\partial} + Q$ is also Γ -equivariant. So we can take Γ -invariants to get a $\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$ -module structure on the complex $[\Phi^\tau(\mathcal{L}_u(\alpha))]^\Gamma$. This construction can also be generalized straightforwardly to twisted complexes of \mathcal{O}^ω of finite rank, i.e. objects of $\text{tw}(\mathcal{O}^\omega)$.

Thus we have describe a functor from $\text{tw}(\mathcal{O}^\omega)$ to $\text{Tw}(\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}})$ on the level of objects. The capital Tw stands for twisted complexes of possibly infinite rank. This is necessary for us here. We denote this functor by $\Phi^\mathcal{P}$ where we think of $\mathcal{P} := [\mathbf{K}^\tau]^\Gamma$ as a certain quantized relative Poincaré bundle.

We continue to define $\Phi^\mathcal{P}$ on morphisms. In fact we will construct $\Phi^\mathcal{P}$ as an A_∞ functor $\text{tw}(\mathcal{O}^\omega) \rightarrow \text{Tw}(\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}})$. Indeed by Appendix B the A_∞ functor Φ^τ has the form

$$\Phi^\tau(a_1, \dots, a_k)(x) := \sum_{l \geq 0, i_0 \geq 0, \dots, i_k \geq 0} m_{l+k+1+i_0+\dots+i_k}(\tau^l, x, \theta^{i_0}, a_1, \theta^{i_1}, \dots, a_k, \theta^{i_k})$$

which by Lemma 5.3 (and its following remark) is also Γ -equivariant. Hence we can also define $\Phi^\mathcal{P}$ as an A_∞ functor.

5.4. Mirror dual of torus fibers. Since the image of $\Phi^\mathcal{P}$ are twisted complexes of infinite rank over $\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$, it is *a priori* unclear whether these objects are quasi-isomorphic to an object in the bounded derived category of coherent sheaves on $M^\vee(\mathbf{U})$ (with coefficients in Λ^π). We show this is the case for Lagrangian torus fibers endowed with a unitary line bundle $[\alpha]$ on it. Here the α is a connection one form corresponding to the line bundle $[\alpha]$ which is determined up to translation by elements of Γ .

5.5. Proposition. *Assume that strictly negative Maslov index does not contribute to structure maps m_k , and assume further that the potential function $W = 0$. Then the object $\Phi^\mathcal{P}(\mathcal{L}_u(\alpha))$ is quasi-isomorphic to the skyscraper sheaf $\Lambda^\pi(u, \alpha)[d]$ over the point $u \in \mathbf{U}$. As in Proposition 4.8 we let an element $f \in \mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$ act on Λ^π via multiplication by $f(u, \alpha)$.*

Proof. The proof is analogous to that of Proposition 4.8. Let us trivialize $M(\mathbf{U}) = \mathbf{T} \times \mathbf{U}$ which induces an identification

$$\Phi^\mathcal{P}(\mathcal{L}_u(\alpha)) \cong [(C_{V^\vee}^\infty \otimes \Omega_u^*) \otimes \Omega^*(\mathbf{T})]^\Gamma.$$

This complex is endowed with the Γ -equivariant differential $\bar{\partial} + Q$ as describe above. Moreover we have $[\bar{\partial}, Q] = Q^2 = 0$ as before. To calculate the cohomology of this complex we first observe a quasi-isomorphism

$$F_1 : ([\mathcal{A}(\mathbf{T}\mathbf{U}) \otimes \Omega^*(\mathbf{T})]^\Gamma, Q) \rightarrow [(C_{V^\vee}^\infty \otimes \Omega_u^*) \otimes \Omega^*(\mathbf{T})]^\Gamma, \bar{\partial} + Q$$

where recall that $\mathcal{A}(\mathbf{T}\mathbf{U})$ is Λ^π valued holomorphic function on $\mathbf{T}\mathbf{U}$. The map F_1 is defined by

$$[(f \cdot T^\beta) \otimes \zeta] \mapsto (e^{\sum_i \langle \partial\beta, e_i \rangle x_i^\vee} \cdot f \cdot T^\beta) \otimes \zeta$$

where again the extra term $e^{\sum_i \langle \partial\beta, e_i \rangle x_i^\vee}$ appears due to the non-trivial action of $\bar{\partial}$ on T^β 2.1. That F_1 is a quasi-isomorphism follows from the exactness of Dolbeault complex.

Next we define a morphism of $\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$ -modules

$$F_2 : ([\mathcal{A}(\mathbf{T}\mathbf{U}) \otimes \Omega^*(\mathbf{T})]^\Gamma, Q) \rightarrow \Lambda^\pi(\mathbf{u}, \alpha)[d]$$

by formula $F(f \otimes \zeta) := f(\mathbf{u}, \alpha) \int_{\mathbf{T}} \zeta$. It is clear that F_2 respects the action of $\mathcal{O}_{M^\vee(\mathbf{U})}^{\text{hol}}$ on both sides. Let us check that F_2 is a morphism of complexes, i.e. we would like to show that $F \circ Q = 0$. For this observe that integration on \mathbf{T} kills all elements of form degree strictly less than d on \mathbf{T} . Moreover by the Maslov index assumption the only operator that increases this degree is Q_0 corresponding to trivial holomorphic disks. As in Proposition 4.8 this operator is explicitly given by

$$\begin{aligned} Q_0(f \otimes \zeta) &= f \otimes d_{\text{dR}} \zeta + m_{2,0}(\tau, f \otimes \zeta) + m_{2,0}(f \otimes \zeta, \theta) \\ &= f \otimes d_{\text{dR}} \zeta + \sum_i (x_i^\vee + \sqrt{-1} y_i^\vee - u_i - \sqrt{-1} \alpha_i) \cdot f \otimes e_i \wedge \zeta. \end{aligned}$$

Applying F_2 to this sum, which by definition is evaluation at (\mathbf{u}, α) and integrate over \mathbf{T} , yields zero. Thus we have shown that F_2 is a map of complexes. It remains to prove that it is also a quasi-isomorphism. For this we consider the spectral sequences associated to energy filtration on both sides. As in the proof of Proposition 4.8 it suffices to analyze the case for $\beta = 0$. This is done in the following lemma, which finishes the proof the proposition. \square

5.6. Lemma. *Denote by $\mathcal{A}(\mathbf{T}\mathbf{U}, \mathbb{C})$ the sheaf of holomorphic functions on $\mathbf{T}\mathbf{U}$ with values in \mathbb{C} , then the map*

$$\varphi : ([\mathcal{A}(\mathbf{T}\mathbf{U}, \mathbb{C}) \otimes \Omega^*(\mathbf{T})]^\Gamma, Q_0) \rightarrow \mathbb{C}[d]$$

defined by $\varphi(f \otimes \zeta) := f(\mathbf{u}, \alpha) \cdot \int_{\mathbf{T}} \zeta$ is a quasi-isomorphism.

Proof. This is a classical result in Fourier transform of a family. We include a proof here for completeness. The proof is similar to that of Proposition 2.6, 2.7 and 2.8 in [3]. We only do this for the case when the dimension of \mathbf{T} is one, i.e. \mathbf{T} is a circle. The general case follows from Kunneth type argument.

We work in the universal cover of \mathbf{T} with affine coordinate \mathbf{y} . Coordinates on $\mathbf{T}\mathbf{U}$ are x^\vee and y^\vee . The operator Q_0 acts by $[\partial/\partial \mathbf{y} + (z^\vee - u - \sqrt{-1}\alpha)]d\mathbf{y}$ where $z^\vee = x^\vee + \sqrt{-1}y^\vee$. To analyze the cohomology of Q_0 it is better to work in another “gauge”, i.e. we conjugate the operator Q_0 with an automorphism which is given by multiplication by $e^{(z^\vee - u - \sqrt{-1}\alpha) \cdot \mathbf{y}}$. Under this conjugation the operator Q_0 is identified with $\partial/\partial \mathbf{y} \circ d\mathbf{y}$, the de Rham differential in \mathbf{y} -direction. Moreover the conjugation also changes the lattice group action on the variables \mathbf{y} and y^\vee . In the y^\vee -direction $\Gamma \subset \mathbb{R}^\vee$ acts simply by translation, while in the \mathbf{y} -direction $\mathbf{n} \in \mathbb{Z} \subset \mathbb{R}$ acts on $s \in C^\infty(\mathbb{R} \times \mathbb{R}^\vee)$ by

$$s(\mathbf{y}, y^\vee) \mapsto e^{n \cdot (z^\vee - u - \sqrt{-1}\alpha)} s(\mathbf{y} - \mathbf{n}, y^\vee).$$

To prove the lemma it suffices to show that an element $f(z^\vee) \otimes s(\mathbf{y})d\mathbf{y}$ is exact if and only if $f(\mathbf{u}, \alpha) \int_0^1 s(v)dv = 0$. The element $f \otimes s d\mathbf{y}$ is exact if there exists an anti-derivative of the form

$$f(z^\vee) \otimes t(\mathbf{y}) := f(z^\vee) \otimes \int_0^{\mathbf{y}} s(v)dv + h(z^\vee)$$

which is periodic in y^\vee -direction and in \mathbf{y} -direction we have

$$f(z^\vee) e^{n \cdot (z^\vee - u - \sqrt{-1}\alpha)} \otimes t(\mathbf{y} - \mathbf{n}) = f(z^\vee) \otimes t(\mathbf{y}).$$

Using the fact that $f \otimes s$ is lattice group invariant we find that such an anti-derivative exists if and only if

$$(e^{n \cdot (z^\vee - u - \sqrt{-1}\alpha)} - 1)h = f \int_0^n s(v) dv$$

for all n . Denote by $q := e^{z^\vee - u - \sqrt{-1}\alpha}$, then using the lattice group invariance of $s(v)$ we get $\int_0^n s(v) dv = (1 + q + \dots + q^{n-1}) \int_0^1 s(v) dv$.

If $f(u, \alpha) \int_0^1 s(v) dv = 0$, then either $f(u, \alpha) = 0$ or $\int_0^1 s(v) dv = 0$. In the first case the fraction

$$\frac{f \int_0^n s(v) dv}{q^n - 1} = \frac{f \int_0^1 s(v) dv}{q - 1}$$

which is independent of n extends to the point (u, α) since $(e^{n \cdot (z^\vee - u - \sqrt{-1}\alpha)} - 1)$ vanishes in first order at (u, α) . If $\int_0^1 s(v) dv = 0$, then $\int_0^n s(v) dv = 0$. Hence we can take h to be simply zero. So in either case the form $f \otimes s dy$ is exact.

Conversely if $f \otimes s dy$ is exact then $f(u, \alpha) \int_0^n s(v) dv = 0$ of all n . In particular we have $f(u, \alpha) \int_0^1 s(v) dv = 0$. The lemma is proved. \square

Theorem 4.10 in the previous section can also be generalized to this situation for \mathcal{O}^ω and $\mathcal{O}_{M^\vee(u)}^{\text{hol}}$. This result is summarized in the following theorem. Its proof is again to use spectral sequences associated to energy filtrations to reduce to classical results. In this case instead of using classical Koszul duality we use classical Fourier transform for families, see for instance [3]. We shall not repeat the proof here.

5.7. Theorem. *The composition of A_∞ functors $\text{Fuk}^\pi(M) \xrightarrow{P} \text{tw}(\mathcal{O}^\omega) \xrightarrow{\Phi^\mathcal{P}} \text{Tw}(\mathcal{O}_{M^\vee(u)}^{\text{hol}})$ is a homotopy equivalence onto its image. Here the first functor P was defined in Theorem 3.1.*

Remark: Here we need to use $\text{Tw}(\mathcal{O}_{M^\vee(u)}^{\text{hol}})$ to include infinite rank objects over $\mathcal{O}_{M^\vee(u)}^{\text{hol}}$. It is an interesting question to do homological perturbation on $\Phi^\mathcal{P} \circ P(\mathcal{L}_u(\alpha))$ to reduce to an object of finite rank. This problem is related to the appearance of theta functions in mirror symmetry.

6. Homological mirror symmetry on toric manifolds

As an immediate application of our general theory we prove a version of homological mirror symmetry between a toric symplectic manifold and its Landau-Ginzburg mirror.

6.1. Theorem. *Let M be a compact smooth toric symplectic manifold, and denote by $\pi : M(\Delta^{\text{int}}) \rightarrow \Delta^{\text{int}}$ the Lagrangian torus fibration over the interior of the polytope of M . Then there exists an A_∞ functor $\Psi : \text{Fuk}^\pi(M) \rightarrow \text{tw}(\mathcal{O}_{\Delta^{\text{int}}}^{\text{hol}})$ which is a homotopy equivalence onto its image.*

Proof. It follows from the results in [11] that with the canonical integrable complex structure on M we already have a sheaf of A_∞ algebras over Δ^{int} , and that the weak unobstructedness assumption (see Section 3) holds. Thus all results in this paper applies to this situation, and the theorem is simply an example of Theorem 4.10. \square

Remark: The functor Ψ in the above theorem is simply the composition $\Phi^\tau \circ P$ where P is the propagation functor used in Theorem 3.1, and Φ^τ is the Koszul duality functor Φ^τ defined

in Section 4. The image of Ψ consists of matrix factorizations of W on $T\Delta^{\text{int}}$ which can be calculated by the formula

$$Q(f \otimes e_I) := \sum_{k \geq 0, l \geq 0} m_{k+l+1}(\tau^l, f \otimes e_I, \theta^k).$$

It is plausible that these objects split generate $\text{tw}(\mathcal{O}_{T\Delta^{\text{int}}}^{\text{hol}})$. But we do not know how to prove this generation result in general. Some of the difficulties are

- working over Novikov ring instead of \mathbb{C} ;
- the map Q mixes various wedge degrees.

In the Fano case we can specialize to $T = e^{-1}$ and work over \mathbb{C} , which enables us to get around the first issue. When the dimension is less than or equal to two the inhomogeneity does not appear, which allows us to prove the following.

6.2. Theorem. *Let M be a compact smooth toric Fano symplectic manifold of dimension less or equal to three. In this case we can work over \mathbb{C} by evaluating the parameter T at e^{-1} . Then there is a functor $\Psi^{\mathbb{C}} : \text{Fuk}^{\pi}(M, \mathbb{C}) \rightarrow \text{tw}(\mathcal{O}_{T\Delta^{\text{int}}}^{\text{hol}} \otimes \mathbb{C})$ which is a quasi-equivalence of A_{∞} categories.*

Proof. The functor $\Psi^{\mathbb{C}}$ is simply the reduction of Ψ at the evaluation $T = e^{-1}$, which is valid under the Fano condition. By the previous theorem it suffices to show that the image of $\Psi^{\mathbb{C}}$ split generates the target category $\text{tw}(\mathcal{O}_{T\Delta^{\text{int}}}^{\text{hol}} \otimes \mathbb{C})$. This generation follows from explicitly computing the operator Q and using the generation result of T. Dyckerhoff [6] Section 4. Note that this generation result requires W to have isolated singularities which was proved in [11] Theorem 10.4.

Next we compute the operator Q . In the following the degree $|I|$ of e_I is referred to as wedge degree. Recall the operator Q_{β} is of wedge degree $1 - \mu(\beta)$. Since for toric manifolds there are no negative Maslov index holomorphic disks, the operator $Q = \sum_{\beta \in G} Q_{\beta}$ has only one part Q_0 that increases the wedge degree. As we saw in the proof of Proposition 4.8 Q_0 is the Koszul differential associated to the regular sequence $z_i^{\vee} - u_i - \sqrt{-1}\alpha_i$.

If the dimension is less than or equal to two, then the operator Q_{β} is necessary of wedge degree -1 corresponding to $\mu(\beta) = 2$. Such type of matrix factorizations is shown to split generate $\text{tw}(\mathcal{O}_{T\Delta^{\text{int}}}^{\text{hol}} \otimes \mathbb{C})$ by [6] Section 4. \square

6.3. An example: \mathbb{CP}^1 . A particular simple example is the case $M = \mathbb{CP}^1$. Since it is Fano we shall work over \mathbb{C} . With appropriate choice of its symplectic form we assume $U = (0, 1) \subset \mathbb{R}$ as is in [11] Section 5. Let e be a trivialization of $R^1\pi_*\mathbb{Z}$, let x^{\vee} , y^{\vee} , y be associated affine coordinates. It is known that the A_{∞} algebra associated to the Lagrangian torus fiber $L_{x^{\vee}}$ for $x^{\vee} \in (0, 1)$ is a two dimensional vector space generated by $\mathbf{1}, e$ with $\mathbf{1}$ a strict unit. All the rest A_{∞} products are

$$\begin{aligned} m_0 &= \exp(-x^{\vee}) + \exp(x^{\vee} - 1); \\ m_1(e) &= \exp(-x^{\vee}) - \exp(x^{\vee} - 1); \\ &\dots; \\ m_k(e^{\otimes k}) &= \frac{1}{k!}[\exp(-x^{\vee}) + (-1)^k \exp(x^{\vee} - 1)]; \\ &\dots. \end{aligned}$$

The potential function is equal to

$$\begin{aligned}
W(x^\vee, -\sqrt{-1}y^\vee) &= \sum_{i=0}^{\infty} m_k((- \sqrt{-1}y^\vee e)^{\otimes k}) \\
&= \sum_{i=0}^{\infty} \frac{1}{k!} (-\sqrt{-1}y^\vee)^k [\exp(-x^\vee) + (-1)^k \exp(x^\vee - 1)] \\
&= \exp(-z^\vee) + \exp(z^\vee - 1).
\end{aligned}$$

Thus W is indeed holomorphic function. Let $a \in \mathbb{R}$ be a real number, and consider the Lagrangian brane $(L_u, -\sqrt{-1}a)$. From this data we get define an A_∞ module $\mathcal{L}_u(\sqrt{-1}a)$ over $\mathcal{O}_{M(u)}^{\omega, \text{can}}$ with internal curvature $W(u, a)$ by constructions in Section 3. Let us describe its image under the Koszul functor Φ^τ . It suffice to compute the operator Q on generators $\mathbf{1}$ and e . For this we have

$$\begin{aligned}
Q(\mathbf{1}) &= \sum_{k,l} m_{k+l+1}(\tau^l, \mathbf{1}, \theta^k) \\
&= \sum_{k,l} (x^\vee - u - \sqrt{-1}a)^k (-\sqrt{-1}y^\vee)^l m_{k+l+1}(e^{\otimes l}, \mathbf{1}, e^{\otimes k}) \\
&= e \otimes [(x^\vee - u - \sqrt{-1}a) - (-\sqrt{-1}y^\vee)] \quad (\text{by our sign convention}) \\
&= e \otimes [z^\vee - u - \sqrt{-1}a].
\end{aligned}$$

A more technical computation of $Q(e)$ by formula gives $\frac{W-W(u,a)}{u+\sqrt{-1}a-z^\vee}$, and hence $Q^2 + [W - W(u, a)] \text{id} = 0$ as is expected from the general theory. Thus we see that $\Phi^\tau(\mathcal{L}_u(\sqrt{-1}a))$ is a matrix factorization of $-[W - W(u, a)]$.

7. Further discussions

In this last section we comment on how constructions in this paper might be globalized over the smooth locus B^{int} of a Lagrangian torus fibration $M \rightarrow B$. The usage of homotopy theory of sheaves with values in an ∞ -category naturally appears in order to globalize the previous local constructions. More details will appear in a forthcoming paper [28]. However it is not clear how the singular locus B^{sing} should enter into this framework.

7.1. Algebraic version of SYZ proposal. In [27] the authors suggested to geometrically understand the mirror phenomena as a duality between dual torus fibrations. The main point we would like to make may be summarized as the following more algebraic point of view of the SYZ proposal.

Mirror symmetry is a duality between a sheaf of A_∞ algebras and its associated family of Maurer-Cartan moduli spaces.

Namely we think of the mirror construction being obtained by taking fiber-wise (weak and purely imaginary) Maurer-Cartan moduli spaces of a family of A_∞ algebras over a base manifold.

7.2. The case without quantum corrections. Let us see how the above idea works if we assume that for all $u \in \mathcal{U}$ there exists no non-trivial J-holomorphic disks in M with its boundary in L_u . In this case we can work over \mathbb{C} . Then the fiber-wise A_∞ algebras are simply the de Rham algebra $\Omega^*(L_u, \mathbb{C})$ over each point $u \in \mathcal{U}$. Its Maurer-Cartan elements are one forms $\alpha \in \Omega^1(L_u, \mathbb{C})$ such that

$$d\alpha + \alpha \wedge \alpha = 0.$$

Since the second term $\alpha \wedge \alpha$ is always zero, these are simply closed one forms on L_u . On the set of Maurer-Cartan elements acts the gauge group whose quotient space is defined as the Maurer-Cartan moduli space. In our case this boils down to identify Maurer-Cartan elements by relation

$$\alpha \cong \alpha + f^{-1}df$$

for any non-vanishing function on L_u . It is well-known we have the following correspondences

$$\mathrm{MC}(\Omega^*(L_u, \mathbb{C})) \Leftrightarrow \text{rank one local system on } L_u \Leftrightarrow \mathrm{Hom}(\pi_1(L_u), \mathbb{C}^*)$$

Through this correspondence purely imaginary Maurer-Cartan moduli space gets identified with unitary local systems, and with $\mathrm{Hom}(\pi_1(L_u), \mathbb{U}(1)) = L_u^\vee$.

When quantum corrections are not presented this construction can easily globalized. Indeed if V is another small open subset of B^{int} , then over the intersection $\mathcal{U} \cap V$, the Maurer-Cartan moduli spaces are glued according to affine coordinate change from \mathcal{U} to V . These gluing certainly satisfies the cocycle condition, which yields a manifold $M^\vee(B^{\mathrm{int}})$ fibered over B^{int} . This is exactly the original SYZ's geometric picture of mirror duality.

7.3. Quantum corrections. When quantum corrections are presented the situation is much more sophisticated. Let us first consider one small open subset \mathcal{U} in B^{int} . By shrinking \mathcal{U} if necessary and performing the constructions in Section 2, we get a local family of A_∞ algebras on the sheaf $\Omega_\pi(\Lambda^\pi)$ over \mathcal{U} whose A_∞ structure maps are compatible with its natural D-module structure, i.e. we have a differential A_∞ algebra over \mathcal{U} . Let us denote this sheaf of A_∞ algebras over \mathcal{U} by $\mathcal{F}(\mathcal{U})$.

Let V be another such open subset, and let $\mathcal{F}(V)$ denote the corresponding sheaf of A_∞ algebras over V . As a first observation, let us point out that $\mathcal{F}(\mathcal{U})$ and $\mathcal{F}(V)$ do not glue to a sheaf of A_∞ algebras over $\mathcal{U} \cup V$. Over the intersection the best one can hope to have is an A_∞ homotopy between the two A_∞ algebras $\mathcal{F}(\mathcal{U})|_V$ and $\mathcal{F}(V)|_{\mathcal{U}}$. Indeed even the very definition of $\mathcal{F}(V)$ from [7] is up to homotopy equivalence.

7.4. Entering of derived geometry in symplectic geometry. Thus we are in a situation where the homotopy theory of sheaves of algebras has to enter. This kind of homotopy theory is itself an active research subject often referred to as derived geometry, or geometry over model categories (or over ∞ -categories). In our case it is well-known A_∞ algebras are fibrant objects in the Quillen model category of differential graded coalgebras. Then the category of presheaves of A_∞ algebras over a space X also carries a global model category structure which does not remember the topology of X . ‘‘Homotopy sheaves’’ of A_∞ algebras over X are then fibrant objects in this model category such that certain descent condition with respect to hypercovers of X is satisfied. The main result that will be proved in [28] is the following claim, which should be a consequence of the fact that the space of tamed almost complex structures on a symplectic manifold is contractible.

Claim. The local A_∞ algebras $\mathcal{F}(\mathcal{U})$'s can be glued using coherent higher homotopies to obtain a homotopy sheaf of A_∞ algebras over B^{int} .

Now assume that we have a homotopy sheaf \mathcal{F} of A_∞ algebras over B^{int} , then it follows that for any open subset O in B^{int} , and any open covering $O = \cup_i \mathcal{U}_i$ we have a homotopy equivalence between A_∞ algebras

$$\mathcal{F}(O) \cong \varprojlim \left\{ \prod_i \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\mathcal{U}_{ij}) \rightrightarrows \prod_{i,j,k} \mathcal{F}(\mathcal{U}_{ijk}) \cdots \right\}. \quad (7.1)$$

Applying Maurer-Cartan functor to this homotopy equivalence we get an isomorphism of sets

$$\text{MC}(\mathcal{F}(O)) \cong \varprojlim \left\{ \prod_i \text{MC}(\mathcal{F}(\mathcal{U}_i)) \rightrightarrows \prod_{i,j} \text{MC}(\mathcal{F}(\mathcal{U}_{ij})) \rightrightarrows \prod_{i,j,k} \text{MC}(\mathcal{F}(\mathcal{U}_{ijk})) \cdots \right\}$$

which implies the cocycle condition to glue these fiber-wise Maurer-Cartan spaces to obtain the “mirror” manifold over B^{int} .

7.5. Derived mirror symmetry? The weak unobstructedness assumption we introduced in Section 3 was to ensure the smoothness of Maurer-Cartan moduli spaces, which yields a smooth mirror manifold.

In general without such an assumption it is plausible to take what might be called the derived Maurer-Moduli spaces which are simplicial sets. Namely for each an A_∞ algebra (with certain convergence assumption) one can define a simplicial set $\text{MC}^\infty(A)$ by

$$[n] \mapsto \text{MC}(A \otimes \Omega^*(\Delta^n))$$

where $\Omega^*(\Delta^n)$ is algebraic differential forms on the standard n -simplex. Applying this construction fiber-wise to our situation, then equation 7.1 implies a homotopy equivalence

$$\text{MC}^\infty(\mathcal{F}(O)) \cong \varprojlim \left\{ \prod_i \text{MC}^\infty(\mathcal{F}(\mathcal{U}_i)) \rightrightarrows \prod_{i,j} \text{MC}^\infty(\mathcal{F}(\mathcal{U}_{ij})) \rightrightarrows \prod_{i,j,k} \text{MC}^\infty(\mathcal{F}(\mathcal{U}_{ijk})) \cdots \right\}.$$

That is we get a homotopy sheaf of simplicial sets, which suggests a possible generalization of mirror symmetry between a symplectic manifold and a certain derived complex manifold.

7.6. Global symplectic functions and mirror duality. The globalization of the construction of symplectic functions can be approached in similar lines since both the Gauss-Manin connection and the symplectic form ω involved to pass from \mathcal{F} to \mathcal{O}^ω are globally defined objects.

To define a mirror sheaf \mathcal{O}^{hol} globally we observe that since the potential function is a homotopy invariant of A_∞ algebras the potential function W is already a global object. But it is important to prove that the change of coordinates obtained from identifying Maurer-Cartan moduli spaces from homotopies of A_∞ algebras is holomorphic so that we obtain a holomorphic mirror manifold fibered over B^{int} . Then we can define \mathcal{O}^{hol} in the same way as in Section 4.

Once \mathcal{O}^ω and \mathcal{O}^{hol} are constructed globally mirror symmetry can be studied in the way as in Sections 4 and 5.

A. Modules with internal curvature.

Let A be an A_∞ algebra. In this section we define A_∞ modules over A possibly with an internal curvature. We show how weak Maurer-Cartan elements of A give rise to such structures. These algebraic constructions naturally occur in Lagrangian Floer theory.

Throughout the construction we work over a base ring \mathbb{R} , and all modules considered here are free \mathbb{R} -modules. We follow the sign convention used in [9]. We refer to [17] Section 3 and 4 for basics of A_∞ algebras, homomorphisms and modules.

A.1. A_∞ modules. An A_∞ module M over an A_∞ algebra A is defined by a collection of maps $\rho_k(-; -) : (A^{\otimes k}) \otimes M \rightarrow M$ of degree $1 - k$ such that

$$\sum_{i+j=N} \rho_i(\text{id}^i; \rho_j(\text{id}^j; -)) + \sum_{r+s+t=N} \rho_{r+t+1}(\text{id}^r, m_s(\text{id}^s), \text{id}^t; -) = 0 \quad (\text{A.1})$$

for all $N \geq 0$. When applied to elements $(a_1 \otimes \cdots \otimes a_N \otimes x) \in A^{\otimes N} \otimes M$ extra signs come out by Koszul sign rule, for example when $N = 0, 1$ the above relation reads

$$\begin{aligned} \rho_0(\rho_0(x)) + \rho_1(m_0; x) &= 0; \\ \rho_0(\rho_1(a; x)) + (-1)^{|a|-1} \rho_1(a; \rho_0(x)) &+ \\ + \rho_1(m_1(a); x) + \rho_2(m_0, a; x) + (-1)^{|a|-1} \rho_2(a, m_0; x) &= 0. \end{aligned}$$

Using the Bar construction we can interpret an A_∞ module structure on a \mathbb{R} -module M as an A_∞ homomorphism $\rho : A \rightarrow \text{End}(M)$ ¹². Recall an A_∞ homomorphism between This correspondence is explicitly given by

$$\{\rho_k\}_{k=0}^\infty \mapsto \rho := \prod_{k=0}^\infty \rho_k \in \text{Hom}_{A_\infty}(A, \text{End}(M))$$

A.2. A_∞ -modules with internal curvature. We need to deal with a slightly weaker notion of modules: those endowed with “internal curvatures”. To introduce this structure we let $\lambda \in \mathbb{R}$ be an even element in the ground ring.

A.3. Definition. An A_∞ module M over A with internal curvature λ is defined by structure maps $\rho_k(-; -) : (A^{\otimes k}) \otimes M \rightarrow M$ of degree $1 - k$ which satisfies the same axioms as in equation A.1 except for $N = 0$ in which case we require that

$$\rho_0(\rho_0(x)) + \rho_1(m_0; x) = \lambda \text{id}_M.$$

From the point of view of A_∞ homomorphisms, we can add the element λid_M as a curvature element for the matrix algebra $\text{End}(M)$, and we denote the resulting curved algebra by $\text{End}^\lambda(M)$. Then straight-forward computation shows that an A_∞ module M over A with internal curvature λ is the same as an A_∞ homomorphism $\rho : A \rightarrow \text{End}^\lambda(M)$.

¹²Note that the product on the graded matrix algebra $\text{End}(M)$ is defined by $(\varphi \otimes \psi) \mapsto (-1)^{|\varphi|} \varphi \circ \psi$ where the sign appears due to our sign convention.

A.4. From weak Maurer-Cartan elements to modules with internal curvature. Let us see how A_∞ modules with internal curvature can arise from a weak Maurer-Cartan element of A . Recall if A is an A_∞ algebra with a strict unit $\mathbf{1}$, an odd element $\mathbf{b} \in A^1$ is a weak Maurer-Cartan element if we have

$$\sum_{k=0}^{\infty} m_k(\mathbf{b}, \dots, \mathbf{b}) = \lambda \mathbf{1}$$

for some even element $\lambda \in R$. Using such an element we can define an A_∞ module structure on the same underlying space of A which has internal curvature λ . We denote this A_∞ module by $A^{\mathbf{b}}$. Its structure maps are defined by

$$\rho_k^{\mathbf{b}}(a_1, \dots, a_k; x) := \sum_{i=0}^{\infty} m_{i+k+1}(a_1, \dots, a_k, x, \mathbf{b}^{\otimes i})$$

Let us check the first axiom, i.e. $\rho_0^{\mathbf{b}}(\rho_0^{\mathbf{b}}(x)) + \rho_1^{\mathbf{b}}(m_0; x) = \lambda \text{id}_{A^{\mathbf{b}}}$.

$$\begin{aligned} \rho_0^{\mathbf{b}}(\rho_0^{\mathbf{b}}(x)) &= \sum_{i,j} m_{i+j+1}(m_{i+1}(x, \mathbf{b}^{\otimes i}), \mathbf{b}^{\otimes j}) \\ &= - \sum_{r \geq 0, s \geq 0, t \geq 0} (-1)^{|x|-1} m_{r+t+2}(x, \mathbf{b}^{\otimes r}, m_s(\mathbf{b}^{\otimes s}), \mathbf{b}^{\otimes t}) - \sum_{k \geq 0} m_{k+2}(m_0, x, \mathbf{b}^{\otimes k}) \\ &= (-1)^{|x|} m(x, \lambda \mathbf{1}) - \rho_1^{\mathbf{b}}(m_0; x) \quad (\text{by the weak Maurer-Cartan equation}) \\ &= \lambda \text{id} - \rho_1^{\mathbf{b}}(m_0; x) \quad (\text{by our sign convention of strict unit}). \end{aligned}$$

The rest identities can be checked similarly using the fact that $\lambda \mathbf{1}$ is a multiple of strict unit, and hence does not contribute to higher products. We denote by $\rho^{\mathbf{b}} : A \rightarrow \text{End}^\lambda(A^{\mathbf{b}})$ the corresponding A_∞ homomorphism.

A.5. Twisted complexes. The category of A_∞ modules are usually defined as a differential graded category. However for purposes of the current paper it is better to use the category of twisted complexes of A which is an A_∞ category. We refer to the paper of B. Keller [17] Section 7 for details of these categorical constructions. The category of twisted complexes over an A_∞ algebra will be denoted by $\text{tw}(A)$. Intuitively this can be thought of as the A_∞ analogue of differential graded modules over an algebra A that are free of finite rank.

To include modules with internal curvatures we need to modify slightly the definition of $\text{tw}(A)$. In the following we explain this modification. This modified version of $\text{tw}(A)$ is a direct sum of R -linear categories

$$\text{tw}(A) := \coprod_{\lambda \in R^{\text{even}}} \text{tw}^\lambda(A).$$

For each $\lambda \in R^{\text{even}}$ the category $\text{tw}^\lambda(A)$ consists of twisted complexes over A with internal curvature λ . Thus the conventional definition of $\text{tw}(A)$ corresponds to $\text{tw}^0(A)$ in our notation. Let us explain in more detail the construction of $\text{tw}^\lambda(A)$.

For each λ the objects of $\text{tw}^\lambda(A)$ are pairs (V, \mathbf{b}) where V is a finite rank $\mathbb{Z}/2\mathbb{Z}$ -graded free R -module, and \mathbf{b} is a weak Maurer-Cartan element of the tensor product $A \otimes \text{End}_R(V)$ with internal curvature λ . By constructions in the previous paragraph these data give rise to an

A_∞ module over $A \otimes V$ with internal curvature λ . Strictly speaking in the previous paragraph we only dealt with the case when V is of rank one over R , but the general case only requires more index.

Let us illustrate the morphism space between two pairs (V, b) and (W, δ) when both V and W is one dimensional. In this case the Hom space, as a graded R -module, is simply A itself. It is endowed with a differential d twisted by b and δ which is explicitly given by formula

$$a \mapsto \sum_{k,l} m_{k+l+1}(b^k, a, \delta^l).$$

Using A_∞ relations and Maurer-Cartan equations one shows that $d^2 = [F(b) - F(\delta)] \text{id} = [\lambda - \lambda] \text{id} = 0$ where $F(b)$ and $F(\delta)$ are internal curvatures of b and δ . This explains the reason why twisted complexes with different internal curvatures do not interact with each other. For the general case when V and W are of any finite rank, the definition is similar using matrix compositions. We refer the details to [17] Section 8.

A.6. Upper-triangular condition. Finally we end this appendix with an important technical point involved in the construction of twisted complexes. In conventional definitions one usually assumes that the (weak) Maurer-Cartan element $b \in A \otimes \text{End}_R(V)$ to satisfy strict upper-triangular condition. This has two important implications. Namely this assumption implies convergence of Maurer-Cartan equation and also the convergence of twisted differential. Secondly it also implies that the homotopy category of $\text{tw}(A)$ embeds fully into the derived category of A ; moreover the image of this embedding is simply the triangulated closure of A as an A_∞ module over itself (this works when m_0 of A is trivial).

While working with such a condition has nice homological implications, it is too restrictive for applications in mirror symmetry. Indeed it follows from the upper-triangularity that the only rank one twisted complex is A itself if m_0 vanishes. But as is shown in Section 3 we would like to associate to each Lagrangian torus fiber a non-trivial rank one twisted complex. Thus we would like to work with twisted complexes which might not satisfy the upper-triangular condition. In this case convergence of relevant series is not automatic, and needs to be taken care of separately. For Lagrangian Floer theory as needed in this paper this convergence follows from results in [7]. Secondly the homotopy category of $\text{tw}(A)$ in our definition might not admit a fully faithfully embedding into the derived category of A .

B. Koszul duality as Fourier-Mukai transform

In this appendix we construct a Koszul duality functor as a type of affine version of Fourier-Mukai transform. We also define such a functor on modules with internal curvatures. It follows from our definition the Koszul functors preserve internal curvatures of modules.

We will need to use another sign convention [17] since the previous sign convention is not convenient to deal with tensor product of algebras. To avoid possible confusions from using two different signs, we first clarify the relationship between them.

B.1. Sign conventions. In the sign convention used in the previous appendix, the maps m_k are considered as degree one maps between suspensions $(A[1])^{\otimes k} \rightarrow A[1]$. The advantage of doing so is that there is no signs in the A_∞ algebra axioms, i.e. for each $n \geq 0$ we have

$$\sum_{r+s+t=n} m_{r+1+t}(\text{id}^r \otimes m_s \otimes \text{id}^t) = 0.$$

When applying to elements we get signs by the Koszul rule. Using maps m_k we can define its corresponding linear maps $m_k^\epsilon : A^{\otimes k} \rightarrow A$ by requiring the following diagram to be commutative:

$$\begin{array}{ccc} A^{\otimes k} & \xrightarrow{m_k^\epsilon} & A \\ \downarrow [1]^{\otimes k} & & \downarrow [1] \\ (A[1])^{\otimes k} & \xrightarrow{m_k} & A[1]. \end{array}$$

Here the map $[1] : A \rightarrow A[1]$ defined by $a \mapsto a[1]$ is the identity map on the underlying R -module, but since it is a degree one map it yields signs when applied to tensor products by Koszul sign rule. Explicitly we have

$$m_k^\epsilon(a_1, \dots, a_k)[1] = (-1)^{\epsilon_k} m_k(a_1[1], \dots, a_k[1])$$

where $\epsilon_k = \sum_{i=1}^k |a_i|(k-i)$. The above identity applied to the A_∞ axioms of m_k yields

$$(-1)^{\epsilon_n} \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}^\epsilon(\text{id}^r \otimes m_s^\epsilon \otimes \text{id}^t) = 0.$$

Dividing the sign $(-1)^{\epsilon_n}$ gives the A_∞ axioms for the maps m_k^ϵ . Using this relationship between m_k and m_k^ϵ we can freely pass from one to the other. For instance the weak MC equation expressed using m_k^ϵ reads

$$\sum_{k=0}^{\infty} (-1)^{\frac{k(k-1)}{2}} m_k^\epsilon(b^{\otimes k}) = \lambda 1.$$

A strict unit 1 in the ϵ -sign convention becomes

$$\begin{aligned} m_2^\epsilon(1, x) &= m_2^\epsilon(x, 1) = x \quad \text{and} \\ m_k^\epsilon(a_1, \dots, a_i, 1, \dots, a_{k-1}) &= 0 \quad \text{for all } k \neq 2. \end{aligned}$$

B.2. Tensor product. Let A be a strict unital A_∞ algebra, and let B be a curved differential graded algebra. Form their tensor product $B \otimes A$ which is an A_∞ algebra with structure maps defined by

$$\begin{aligned} m_0^\epsilon &:= 1 \otimes m_0^\epsilon + W \otimes 1; \\ m_1^\epsilon(b \otimes a) &:= db \otimes a + (-1)^{|b|} b \otimes m_1^\epsilon(a); \\ m_k^\epsilon(b_1 \otimes a_1, \dots, b_k \otimes a_k) &:= (-1)^{\eta_k} (b_1 \cdots b_k) \otimes m_k^\epsilon(a_1, \dots, a_k) \quad \text{for } k \geq 2. \end{aligned}$$

Here W is the curvature term of B , the two 1 's are units, and the sign in the last equation is $\eta_k = \sum_{i=1}^{k-1} |a_i|(|b_{i+1}| + \dots + |b_k|)$. We have abused the notation m_k^ϵ for both structure maps on A and $B \otimes A$. Since they are applied to different types of elements, no confusion can arise by doing so.

B.3. Koszul duality as Fourier-Mukai transform. Next we describe a construction of a functor $\Phi^\tau : \text{tw}(A) \rightarrow \text{tw}(B)$ associated to a given Maurer-Cartan element $\tau \in B \otimes A$. For simplicity we will assume that A is of finite rank over R . This is for the purpose that Φ^τ lands inside $\text{tw}(B)$, i.e. it is of finite rank over B . If we replace the target category by $\text{Tw}(B)$

consisting of twisted complexes over B of possibly infinite rank then all constructions below still go through.

The intuitive idea to construct such a functor is that the element τ determines an A_∞ module $(B \otimes A)^\tau$ over $B \otimes A$ which can be viewed as a kernel for an integral transform from $\text{tw}(A)$ to $\text{tw}(B)$. To realize this idea we proceed as follows.

Given an A_∞ module M with internal curvature λ , we denote by ρ^M the corresponding A_∞ homomorphism $A \rightarrow \text{End}^\lambda(M)$. For $M \in \text{tw}(A)$ by our assumption that A is of finite rank, it follows that M is also of finite rank. For general M we assume M is of finite rank below.

The map ρ^M induces another A_∞ homomorphism $\rho_B^M : B \otimes A \rightarrow B \otimes \text{End}^\lambda(M)$ by scalar extension to B . Using ρ_B^M we can push forward the given Maurer-Cartan element τ to get a Maurer-Cartan element of $\text{End}^\lambda(M) \otimes B$. Such a Maurer-Cartan element by definition is a twisted complex structure on $B \otimes_R M$ with internal curvature λ . The following Theorem gives a more explicit description of this construction with formulas.

B.4. Theorem. *The maps $(\rho_B^M)_k : (B \otimes A)^k \rightarrow B \otimes \text{End}^\lambda(M)$ defined by*

$$\begin{aligned} (\rho_B^M)_0 &:= 1 \otimes \rho_0^M; \\ (\rho_B^M)_1(b_1 \otimes a_1) &:= b_1 \otimes \rho_1^M(a_1); \\ (\rho_B^M)_k(b_1 \otimes a_1, \dots, b_k \otimes a_k) &:= (-1)^{n_k}(b_1 \cdots b_k) \otimes \rho_k^M(a_1, \dots, a_k) \end{aligned}$$

form an A_∞ homomorphism $\rho_B^M : B \otimes A \rightarrow B \otimes \text{End}^\lambda(M)$. Moreover if $\tau \in B \otimes A$ is a Maurer-Cartan element, its push-forward $Q := (\rho_B^M)_* \tau = \sum_{k=0}^{\infty} (-1)^{\frac{k(k-1)}{2}} (\rho_B^M)_k(\tau^{\otimes k})$ is a Maurer-Cartan element of $B \otimes \text{End}^\lambda(M)$, i.e. we have

$$(\lambda - W) \text{id} + [d, Q] - Q^2 = 0.$$

Here $-W$ is the curvature of B , and d is its differential.

Proof. The proof is straightforward verifications of formulas and keeping track of signs. We omit it here. \square

By the above theorem we described what Φ^τ does on the level of objects. Namely we define $\Phi^\tau(M)$ to be the twisted complex on $B \otimes_R M$ defined by the weak Maurer-Cartan element $(\rho_B^M)_* \tau$. Note that $\Phi^\tau(M)$ and M have the same internal curvature, i.e. we have a map

$$\Phi^\tau : \text{tw}^\lambda(A) \rightarrow \text{tw}^\lambda(B)$$

for each $\lambda \in \mathbb{R}^{\text{even}}$.

Next we describe Φ^τ on the level of morphisms. We can define Φ^τ as an A_∞ functor from $\text{tw}(A)$ to $\text{tw}(B)$. Let us illustrate this for a rank one twisted complex A^b over A with internal curvature λ . The space $\text{End}(A^b)$ is an A_∞ algebra with structure maps m_k^b defined by

$$m_k^b(a_1, \dots, a_k) := \sum_{i_0 \geq 0, \dots, i_k \geq 0} m_{i_0+i_1+\dots+i_k+k}(b^{i_0}, a_1, b^{i_1}, \dots, b^{i_{k-1}}, a_k, b^{i_k}).$$

We need to define an A_∞ homomorphism from $\text{End}(A^b)$ to the differential graded algebra $\text{End}(\Phi^\tau(A^b), \Phi^\tau(A^b))$. Since $\Phi^\tau(A^b) = B \otimes_R A$ as a R -module, we use structure maps m_k on the tensor product $B \otimes A$ to describe this homomorphism. Explicitly it is given by

$$\Phi^\tau(a_1, \dots, a_k)(x) := \sum_{l \geq 0, i_0 \geq 0, \dots, i_k \geq 0} m_{l+k+1+i_0+\dots+i_k}(\tau^l, x, b^{i_0}, a_1, b^{i_1}, \dots, a_k, b^{i_k}).$$

The case of higher rank twisted complexes requires no more than putting more index into the above equation. We refer to Section 7.3 of K. Lefèvre-Hasegawa's thesis [20] for a more detailed discussion of this A_∞ homomorphism.

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